Qualifying Exam: Algebra I, Spring 2016

1. (15 points)

   (a) Let X be any set and G be a group acting transitively on X. Show that there is a bijective map from X to G/G_x, denoted as $X \cong G/G_x$, where $G_x := \{h \in G \mid h \cdot x = x\}$ is the stabilizer of some point $x \in X$.

   (b) Let $\mathcal{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$. Show that

   $$\mathcal{H} \cong SL_2(\mathbb{R})/SO(2)$$

   where $SO(2) := \{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \mid a^2 + b^2 = 1\}$.

2. (25 points) Show that the following: a complete set of coser representatives for doule coset

   $$SL_2(\mathbb{Z})\backslash GL_2(\mathbb{Q})^+/SL_2(\mathbb{Z})$$

   consists fo the diagonal matrices

   $$\left(\begin{smallmatrix} d_1 & \\ & d_2 \end{smallmatrix}\right), \text{ where } d_1, d_2 \in \mathbb{Q} \text{ and } \tfrac{d_1}{d_2} \text{ is a positive integer}$$

   Here,

   $$SL_2(\mathbb{Z}) := \{A \in M_{2\times 2}(\mathbb{Z}) \mid \det(A) = 1\}$$

   and

   $$GL_2(\mathbb{Q})^+ := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_{2\times 2}(\mathbb{Q}) \mid ad - bc > 0\}$$

   (Hint: use the elementary divisor theorem: let R be a principal ideal domain. Let Λ_1 be a free R-module of rank n and let Λ_2 be a free R-submodule of rank n. There exist a basis $x_1, .., x_n$ of Λ_1 and nonzero elements $D_1, .., D_n$ of R such that $D_{i+1}|D_i$ for $1 \le i < n$ and such that $D_1 x_1, .., D_n x_n$ is a basis of Λ_2.)

3. (20 points) Prove the following statements:

   (a) Every nonzero prime ideal in a Principal Ideal Domain is a maximal ideal.

   (b) If R is a commutative ring such that $R[x]$ is a Principal Ideal Domain, then R is a necessarily a field.

4. (20 points)

   (a) Show that the quadratic integer ring $\mathbb{Z}[\sqrt{-5}]$ is not a Principal Ideal Domain.

   (b) Find an example of a ring R in whch an irreducible element may not be a prime element (with a verification).

5. (20 points)

   (a) Let p be a prime. Let G be a finite group such that $n_p(G)$ is not congruent to $1 \mod p^2$. Here $n_p(G)$ is the number of distinct Sylow p-subgroups of G. Show that there are Sylow p-subgroups P and R of G such that $P \cap R$ is of index p in both P and R.

   (b) Show that there are no simple groups of order 1053.