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Lemma 1 *The following identity holds for every α in $(-1, 0)$.*[1]

$$\mathcal{F}\left\{|t|^\alpha\right\} = \frac{\Gamma\left(\dfrac{1+\alpha}{2}\right)}{\pi^{\frac{1}{2}+\alpha}\Gamma\left(\dfrac{-\alpha}{2}\right)}|\nu|^{-\alpha-1}$$

Proof

$$\begin{aligned}
\mathcal{F}\{|t|^\alpha\} &= \int_{-\infty}^{\infty} |t|^\alpha e^{-2\pi i\nu t}dt \\
&= \int_{-\infty}^{\infty} |t|^\alpha \cos(-2\pi\nu t)dt \quad (\because |t|^\alpha \text{ is an even function.}) \\
&= 2\int_0^{\infty} t^\alpha \cos(-2\pi|\nu|t)dt \quad (\because \cos \text{ is an even function.}) \\
&= \frac{2}{(2\pi|\nu|)^{\alpha+1}}\int_0^{\infty} s^\alpha \cos s\, ds \quad (\text{where } s = 2\pi i|\nu|t) \\
&= \frac{2}{(\alpha+1)(2\pi|\nu|)^{\alpha+1}}\int_0^{\infty} \cos r^{\frac{1}{\alpha+1}}\, dt \quad (\text{where } r = s^{\alpha+1}) \\
&= \frac{2}{(\alpha+1)(2\pi|\nu|)^{\alpha+1}} \cdot \frac{2^\alpha\sqrt{\pi}\Gamma\left(\dfrac{\alpha+1}{2}\right)}{\dfrac{1}{\alpha+1}\Gamma\left(-\dfrac{\alpha}{2}\right)} \\
&= \frac{\Gamma\left(\dfrac{1+\alpha}{2}\right)}{\pi^{\frac{1}{2}+\alpha}\Gamma\left(\dfrac{-\alpha}{2}\right)}|\nu|^{-\alpha-1}
\end{aligned}$$

[1] $\mathcal{F}$ is the Fourier transform that is written in terms of ordinary frequency.

Lemma 2 *For every α in $(-1,0)$ a convolution root of $|t|^\alpha$ exists, and that can be represented as following.*[2]

$$|t|^\alpha = |kt|^{\frac{\alpha-1}{2}} * |kt|^{\frac{\alpha-1}{2}} \quad \left(\text{where } k = \frac{\Gamma\left(\frac{1-\alpha}{4}\right)\sqrt{\Gamma\left(\frac{\alpha+1}{2}\right)}}{\pi^{\frac{1}{4}}\Gamma\left(\frac{\alpha+1}{4}\right)\sqrt{\Gamma\left(-\frac{\alpha}{2}\right)}} \right)$$

Proof

$$\begin{aligned}
|\nu|^{-(\alpha+1)} &= |\nu|^{-\frac{\alpha+1}{2}} \cdot |\nu|^{-\frac{\alpha+1}{2}} \\
\mathcal{F}\left\{ \frac{\pi^{\frac{1}{2}+\alpha}\Gamma\left(\frac{-\alpha}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2}\right)}|t|^\alpha \right\} &= \mathcal{F}\left\{ \frac{\pi^{\frac{\alpha}{2}}\Gamma\left(\frac{1-\alpha}{4}\right)}{\Gamma\left(\frac{\alpha+1}{4}\right)}|t|^{\frac{\alpha-1}{2}} \right\}^2 \text{ (by Lemma 1)} \\
\mathcal{F}\left\{|t|^\alpha\right\} &= \mathcal{F}\left\{ k^2|t|^{\frac{\alpha-1}{2}} \right\}^2 \\
\mathcal{F}^{-1}\left\{\mathcal{F}\left\{|t|^\alpha\right\}\right\} &= \mathcal{F}^{-1}\left\{ \mathcal{F}\left\{ |kt|^{\frac{\alpha-1}{2}} \right\}^2 \right\} \\
|t|^\alpha &= |kt|^{\frac{\alpha-1}{2}} * |kt|^{\frac{\alpha-1}{2}}
\end{aligned}$$

Lemma 3 *For every α in $(-1,0)$, a continuous function $f : \mathbb{R} \to \mathbb{C}$ satisfy the following integral equation if and only if f is constantly zero.*

$$\int_{-\infty}^{\infty} f(y)|x-y|^\alpha dy = 0$$

Proof

It is easy to verify that if f is constantly 0, f satisfy the given equation. Now, let's prove the other side of the lemma.

Integral equation $\int_{-\infty}^{\infty} f(y)|x-y|^\alpha dy = 0$ is a Fredholm equation of the first kind where the kernel is of the form K(x-y). Also, Fourier transform of both $|t|^\alpha$, 0 are exist. Hence, the solution of the given integral equation is

$$\begin{aligned}
f(y) &= \mathcal{F}^{-1}\left\{ \frac{\mathcal{F}\{0\}}{\mathcal{F}\{|t|^\alpha\}} \right\} \\
&= \mathcal{F}^{-1}\{0\} \\
&= 0.
\end{aligned}$$

Problem 1 *Let $f : \mathbb{R} \to \mathbb{C}$ be a complex valued continuous function on $\mathbb{R}$. For a real number $\alpha \in (-1,0)$, defined*

$$\Phi(f)(x) := \int_{-\infty}^{\infty} f(y)|x-y|^\alpha dy.$$

[2]Convolution operator * is defined as $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$.

This is pattered after the Coulomb potential. Suppose that the corresponding energy is bounded:

$$\frac{1}{2}\int_{-\infty}^{\infty}|f(x)|\Phi(|f|)(x)dx < \infty.$$

(This guarantees the change of the order of integrations.) Then it attains a nonnegative real value. Show that

$$\int_{-\infty}^{\infty}\overline{f(x)}\Phi(f)(x)dx \geq 0,$$

and that the equality holds if and only if $f \equiv 0$.

Proof

$$\begin{aligned}
& \int_{-\infty}^{\infty}\overline{f(x)}\Phi(f)(x)dx \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{f(x)}f(y)|x-y|^{\alpha}dydx \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{f(x)}f(y)|kx|^{\frac{\alpha-1}{2}} * |ky|^{\frac{\alpha-1}{2}}dydx \\
& \text{(by Lemma 2)} \\
= & k^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{f(x)}f(y)|t|^{\frac{\alpha-1}{2}}|(x-y)-t|^{\frac{\alpha-1}{2}}dtdydx \\
= & k^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{f(x)}f(y)|x-\tau|^{\frac{\alpha-1}{2}}|y-\tau|^{\frac{\alpha-1}{2}}d\tau dydx \\
& \text{(where } \tau = t+y\text{)} \\
= & k^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{f(x)}f(y)|x-\tau|^{\frac{\alpha-1}{2}}|y-\tau|^{\frac{\alpha-1}{2}}dydxd\tau \\
& \text{(by fubini's theorem)} \\
= & k^2\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\overline{f(x)}|x-\tau|^{\frac{\alpha-1}{2}}dx\right)\left(\int_{-\infty}^{\infty}f(y)|y-\tau|^{\frac{\alpha-1}{2}}dy\right)d\tau \\
= & k^2\int_{-\infty}^{\infty}\overline{F(\tau)}F(\tau)d\tau \quad \text{(where } F(x) = \int_{-\infty}^{\infty}f(x)|x-\tau|^{\frac{\alpha-1}{2}}dx\text{)} \\
\geq & \ 0
\end{aligned}$$

Also by Lemma 3, F(x) is constantly 0 if and only if f(x) is zero. Thus, the equality holds if and only if f(x) is constantly zero.