Real Analysis Qualifying Exam

2017.01.05

(**Show every detail. You may use the results of previous sub-problems.)**

1. Let $(X, \mathcal{B}, \nu)$ be a σ-finite measure space.

1.1. (10 points) Show that $\mu^*(A) := \inf\{\nu(B) : A \subseteq B,\ B \in \mathcal{B}\}$ defines an outer measure on X.

1.2. (10 points) Show that the measure produced from μ^* by Carathéodory's construction agrees with ν on $\mathcal{B}$. Deduce that X can be extended to a complete measure space.

2.1. (5 points) Let $f, g: \mathbb{R}^n \to \mathbb{C}$ be Lebesgue measurable, $f \in L^1$, $g \in L^\infty$. Show that $f * g$ is well defined, bounded and continuous.

2.2. (5 points) Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set of positive measure, Show that

$$E - E := \{x - y:\ x, y \in E\}$$

contains an open neighborhood D of the origin. [Consider $\chi_E * \chi_{-E}$.]

2.3. (5 points) Let $f: \mathbb{R}^n \to \mathbb{C}$ be a measurable function satisfying

$$f(x + y) = f(x) + f(y), \quad x, y \in \mathbb{R}^n.$$

Show that f is continuous. [Show first that for any disk centered at the origin, $f^{-1}(z + D)$ has a positive measure for at least one $z \in \mathbb{C}$.]

2.4. (5 points) Show that

$$f(x_1, x_2, \cdots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n, \quad \text{for some } c_1, \cdots c_n \in \mathbb{C}$$

3. (10 points) Show that if f is absolutely continuous on $[a, b]$ and $f'(x) = 0$ for almost every $x \in [a, b]$, then f is constant. [Let $c \in [a, b]$, $E := \{x \in [a, c], f'(x) = 0\}$. Use Vitali covering theorem to show $f(a) = f(c)$.]

4. (10 points) Suppose that $f_k, f \in L^2$ and that $\int f_k g\, d\mu \to \int fg\, d\mu$ for all $g \in L^2$. If $\|f_k\|_2 \to \|f\|_2$, show that

$$f_k \to f \text{ in } L^2.$$

5. Suppose that $(X, \mathcal{M}, \mu)$ be a measure space and $1 < p \le \infty$. Let $f \in L^{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$ be given. Define a linear functional $F: L^p \to \mathbb{R}$ by

$$F(g) = \int_X fg\, d\mu, \quad \forall g \in L^p.$$

5.1. (10 points) Show that

$$\|F\| := \sup\{|F(g)| : \|g\|_p \leq 1\} = \|f\|_p,$$

5.2. (10 points) Suppose $p = 1$, and X is σ-finite. Show that $\|F\| = \|f\|_\infty$.

[Given $\epsilon > 0$, let $A = \{x \in X : |f(x)| > \|f\|_\infty - \epsilon\}$. Then $\mu(A) > 0$. Find $B \subset A$ with $0 < \mu(B) < \infty$. Build $g \in L^1$, $\|g\|_1 = 1$ so that $|F(g)| \geq \|f\|_\infty - \epsilon$.]

6. Let $f \in L^1(\mathbb{R}^d)$ and define

$$f^*(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f| \, d\mu$$

where B denotes a ball. Let $1 < p \leq \infty$. [You may use the fact: $\forall \alpha > 0$, $\mu(f^* > \alpha\}) \leq (c/\alpha)\|f\|_1$.] Define

$$g(x) = \begin{cases} f(x), & \text{if } |f(x)| \geq \alpha/2 \\ 0, & \text{otherwise} \end{cases}$$

6.1. (5 points) Show that

$$f^*(x) \leq g^*(x) + \frac{\alpha}{2}$$

6.2. (5 points) Show that for some constant c independent of f,

$$\mu\{f^* > \alpha\} \leq \frac{2c}{\alpha} \int_{|f| \geq \frac{\alpha}{2}} |f(x)| \, d\mu$$

6.3. (10 points) Show that that if $f \in L^p$, then $f^* \in L^p$. [Suggestion: $\int_{\mathbb{R}^d} f^{*p} \, d\mu = \int_0^\infty p\alpha^{p-1} \mu\{f^* > \alpha\} d\alpha$.]