Qualifying Exam **Numerical Analysis** **January, 2017**

1. (20 points) Consider the problem of solving the polynomial equation

$$x^3 - 6x^2 + 11x - 6 = 0,$$

which is from the equation $(x-1)(x-2)(x-3) = 0$. Assuming only the coefficient of x^2 has small error in expanding the product in descending order, we have the equation

$$x^3 - ax^2 + 11x - 6 = 0, \tag{1}$$

where $a \approx 6$. Note that when $a = 6$, the equation has three zeros $x = 1, 2, 3$.

Let $x_1(a)$ be the zero of the perturbed equation (1) such that $x_1(6) = 1$.

(a) Compute the condition number of $x_1(a)$ at $a = 6$, denoted by $cond(x_1(6))$.

(b) By using the result of (a), find an upper bound m of the absolute value of the relative error of $x_1(5.64)$ to $x_1(6)$.

2. (20 points)

(a) Let x_1, x_2, x_3 distinct points. Let $L_{2,k}(x)$ $(k = 1, 2, 3)$ be the Lagrange interpolating polynomial of degree 2 such that for $i = 1, 2, 3$,

$$L_{2,k}(x_i) = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{if } i \neq k. \end{cases}$$

Prove that

$$L_{2,1}(x) + L_{2,2}(x) + L_{2,3}(x) = 1 \quad \text{for all } x.$$

(b) Let $x_1, x_2, \ldots, x_{n+1}$ be distinct points. Let $L_{n,k}(x)$ $(1 \leq k \leq n)$ be the Lagrange interpolating polynomial of degree n. Prove that

$$\sum_{k=1}^{n+1} L_{n,k}(x) = 1 \quad \text{for all } x \text{ and for each positive integer } n \geq 2.$$

3. (20 points) A natural cubic spline S is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + B(x-1) - D(x-1)^3, & \text{if } 1 \leq x < 2, \\ S_1(x) = 1 + b(x-2) - \frac{3}{4}(x-2)^2 + d(x-2)^3, & \text{if } 2 \leq x \leq 3. \end{cases}$$

If S interpolates the data $(1,1)$, $(2,1)$, and $(3,0)$, find B, D, b, and d.

4. (20 points) Find the degree of precision (or accuracy) of the quadrature rule

$$\int_{-1}^{1} f(x)dx \approx f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right).$$

5. (20 points) Estimate the eigenvalues of the following matrix A as accurately as possible:

$$A = \begin{bmatrix} 1 & 0.001 & 0.0001 \\ 0.001 & 2 & 0.001 \\ 0.0001 & 0.001 & 3 \end{bmatrix}.$$

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