1. [20] Let $X_1, \ldots, X_n$ be a random sample from a population with probability density function

$$f(x) = \frac{1}{\pi} \frac{1}{1 + (x - \theta)^2}, \quad -\infty < x < \infty.$$

(a) [10] Obtain the characteristic function of X_1. (Hint: $\displaystyle\int_0^\infty \frac{\cos(tx)}{1+x^2}\, dx = \frac{\pi}{2} e^{-|t|}$.)

(b) [10] Show that $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ follows the same distribution as X_1.

2. [20] Suppose that $X_1, \ldots, X_m$, $m > 1$, and $Y_1, \ldots, Y_n$, $n > 1$, be independent random samples from uniform distributions $U(0, \alpha)$ and $U(0, \beta)$, respectively, for $\alpha > 0$, $\beta > 0$.

(a) [10] What is the complete sufficient statistic for α?

(b) [10] Determine the uniformly minimum variance unbiased estimator of $\theta = \alpha/\beta$.

3. [20] Let $X_1, \ldots, X_n$ be a random sample from a population with distribution function

$$F(x; \theta, \tau) = \begin{cases} 1, & x \geq \tau \\ (x/\tau)^\theta, & 0 \leq x \leq \tau, \\ 0, & x \leq 0 \end{cases}$$

for $\theta > 0$ and $\tau > 0$.

(a) [10] Find the maximum likelihood estimators (MLE) for θ and τ, and verify that they are consistent.

(b) [10] Suppose that $\tau = 1$, and consider the problem of testing $H_0 : \theta \leq 1$ vs $H_1 : \theta > 1$. Show that the uniformly most powerful test of level α, $0 < \alpha < 1$, is given by

$$\text{“Reject } H_0 \text{ if } -2\sum_{j=1}^n \log X_j < \chi^2_{2n}(1-\alpha),\text{”}$$

where $\chi^2_{2n}(1-\alpha)$ is the upper $1-\alpha$ quantile of χ^2_{2n} distribution.

— **Continued on the next page** —

4. [30] Suppose that X and Y are independent random variables with densities

$$g(x) = \frac{1}{\lambda} e^{-x/\lambda}, x > 0, \quad \text{and} \quad h(y) = \frac{1}{\mu} e^{-y/\mu}, y > 0,$$

respectively, for some $\lambda > 0$ and $\mu > 0$. Let $Z = \min(X, Y)$ and $W = \begin{cases} 1, & \text{if } X < Y, \\ 0, & \text{if } X \geq Y. \end{cases}$

(a) [10] Find the joint density function $f(z, w)$ of Z and W. Show that Z and W are independent of each other.

(b) [10] Now suppose that (Z_i, W_i), $i = 1, 2, \ldots, n$, is a random sample from $f(z, w)$. Find the MLE $\hat{\lambda}$ of λ.

(c) [10] Find the asymptotically distribution of $\sqrt{n}(\hat{\lambda} - \lambda)$ as $n \to \infty$. Construct a likelihood ratio test of asymptotic level $\alpha \in (0, 1)$ for testing $H_0 : \lambda = 1$ vs $H_1 : \lambda \neq 1$.

5. [10] Let $X \sim f(x; \theta)$, $\theta \in \Theta$, and $T = T(X)$ be a sufficient statistic for θ. A statistic ψ is called a test function if $\psi \in [0, 1]$. Show that for every test function $\psi(X)$, there exists a test function $\tilde{\psi}(T)$ such that $E_\theta \tilde{\psi}(T) = E_\theta \psi(X)$ for all $\theta \in \Theta$.

— **The end** —