2016–2 Qualifying Exam
Algebraic Topology

1. (15pts) Show that not all compact Hausdorff topological spaces are homotopy equivalent to a finite CW complex.

2. (15pts) Let $X = \{(x, y, z) \in \mathbb{R}^3 \mid (x^2 + y^2 - 2)(y^2 + z^2 - 1) = 0\}$. Compute $H_i(X)$. (In this problem, a homotopy equivalence between two non-homeomorphic spaces should be verified explicitly.)

3. Suppose that X is a space and $0 \to G' \xrightarrow{f} G \xrightarrow{g} G'' \to 0$ is an exact sequence of abelian groups.

   (a) (10pts) Show that there is a long exact sequence
   $$\cdots \to H_i(X; G') \xrightarrow{f_*} H_i(X; G) \xrightarrow{g_*} H_n(X; G'') \xrightarrow{\delta} H_{i-1}(X'; G') \to \cdots$$
   where $f_*([z \otimes a]) = [z \otimes f(a)]$ and $g_*([z \otimes a]) = [z \otimes g(a)]$.

   (b) (15pts) For the case of $X = \mathbb{R}P^3$ and $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$, compute the homomorphisms f_*, g_* and δ for each i.

4. (20pts) State the definition of $\mathrm{Tor}(A, G)$, using a (short) free resolution, for abelian groups A and G. Show that it is well-defined, independent of the choice of a (short) free resolution.

5. A second countable Hausdorff space X is an n-*manifold* ($n \geq 1$) if every point in X has an open neighborhood homeomorphic to $\mathbb{R}^n$.

   (a) (10pts) For an n-manifold X and $p \in X$, compute $H_i(X, X - \{p\})$.

   (b) (15pts) Show that the one-point compactification of $S^5 \times \mathbb{R}$ is not an n-manifold for any n.