PhD QE on Differentiable manifolds and Lie groups; January 2017

Directions. One needs 60 points or better to pass.

[1] (20 pts) If a smooth manifold admits a transitive smooth action by a Lie group, then show that this manifold is always orientable.

[2] (20 pts) Let M be a $\mathcal{C}^\infty$ smooth manifold with $p \in M$. Denote by D the set of germs at p of $\mathcal{C}^\infty$ functions defined on neighborhoods of p. Prove the following:

(a) D becomes a commutative ring.

(b) The set $\underline{\mathrm{m}} := \{g \in D \colon g(p) = 0\}$ is an ideal.

(c) The tangent space T_pM is linearly isomorphic (in a natural way) to the dual space of the quotient space $\underline{\mathrm{m}}/\underline{\mathrm{m}}^2$, where $\underline{\mathrm{m}}^2$ is the linear subspace of $\underline{\mathrm{m}}$ generated by the products of two elements of $\underline{\mathrm{m}}$.

[3] (20 pts) Show with a precise proof that [2](c) is not true if the manifold is only $\mathcal{C}^k$ smooth with k finite.

[4] (20 pts) Let $V_c = \{(x, y, z, w) \in \mathbb{R}^4 \colon x + cy - z^2 = x^3 + z - c^2w^2 = 0\}$. Determine which values of c imply that V_c is a smooth submanifold of $\mathbb{R}^4$. [Note: Smooth manifolds need not be connected, in this problem.]

[5] (20 pts) For an $n \times n$ matrix A, define e^A to be $e^A = I + \sum_{j=1}^{\infty} \frac{A^j}{j!}$.

(a) Justify this definition.

(b) Show that $\det(e^A) = e^{\text{trace } A}$.