Complex analysis

December 2016

1. (10points) Evaluate the following integral and justify your answer.

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\, dx.$$

2. (10points) Find the Laurent expansion of $\tan(z) = \sum_{n=-\infty}^{\infty} a_n \left(z - \frac{\pi}{2}\right)^n$ about $z = \frac{\pi}{2}$.

3. (10points) Show that every polynomial p of degree at least 1 is surjective (that is, for all $a \in \mathbb{C}$ there exists $z \in \mathbb{C}$ such that $p(z) = a$).

4. (30points) Answer the following questions.

   (a) State Rouche's Theorem.

   (b) State Schwarz's Lemma.

   (c) Suppose f is analytic in the unit disk $|z| < 1$ with $|f(z)| \leq 1$ and $f(0) = 0$. Prove that for any integer $n \geq 1$, $f(z) - 2^n z^n$ has precisely n zeros (counting multiplicity) in the disk $|z| < \frac{1}{2}$.

5. (20points) Let $D = \{z \mid |z| < 1\}$. Suppose that $f : D \to \mathbb{C}$ satisfies $Ref(z) \geq 0$ for all z in D and suppose that f is analytic and not constant. Show that if $f(0) = 1$ then

$$|f(z)| \geq \frac{1 - |z|}{1 + |z|}.$$

6. (20points) Let f be an entire function having the property that

$$f(z + m + ni) = f(z) \qquad (z \in \mathbb{C}, m, n \in \mathbb{Z})$$

   Prove that f is constant.