Real Analysis Qualigying Examination. Summer 2016 POSTECH

(6 problems. Justify all your work.)

1. (1) (10pt) Let $E \subseteq \mathbb{R}$ be a Lebesque measurable set. Show that
$$\lim_{t\to 0} m\big(E \cap (E+t)\big) = m(E),$$
where $E + t := \{x + t : x \in E\}$ and m is Lebesque measure on $\mathbb{R}$.
(2) (10pt) Let $E \subset [0,1]$ be a Lebesque measurable set such that
$$m\big(E \cap (E+t)\big) = \big(m(E)\big)^2, \quad \text{for all } |t| \le 1/2.$$
What is $m(E)$?

2. (20pt) Let (X, μ) be a measure space with a (positive) measure μ. Suppose that $f \in L^p(X,\mu) \cap L^\infty(X,\mu)$ for some $p \in (1,\infty)$. Show that
$$\lim_{q\to\infty} \|f\|_{L^q(X)} = \|f\|_{L^\infty(X)}.$$

3. (20pt) Let (X, μ) be a measure space with a (positive) measure μ and $\mu(X) < \infty$. Suppose that each $f_n (n = 1, 2, \dots)$ and f are measurable functions from X to $\mathbb{R}$. Show that f_n converges to f in measure if and only if every subsequence of $\{f_n\}$ has in turn a subsequence that converges almost everywhere to f. (CAUTION: Do NOT try to do this problem by simply applying Egoroff's theorem. If you want to use that theorem, include a proof to receive credit).

4. (10pt) Show that if $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, then there exist constants $a < \infty$ and $b < \infty$ such that
$$|f(x)| \le a|x| + b.$$

5. (10pt) Let $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous. Show that f maps sets of measure zero to sets of measure zero.

6. Let $f \in L^1$. Define
$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixt}\,dx$$
and for $u > 0$
$$S_u(x) = \int_{-u}^{u} \hat{f}(t)e^{2\pi itx}\,dt.$$
(1) (5pt) If $D_u(x) = \int_{-u}^{u} e^{2\pi itx}\,dt$, then show that
$$D_u(x) = \frac{\sin 2\pi ux}{\pi x} \quad \text{and} \quad S_u(x) = f * D_u(x).$$
(2) (15pt) Suppose that for given $x_0 \in \mathbb{R}$, the following holds:
$$\int_0^1 \left| \frac{f(x_0+y) + f(x_0-y) - 2f(x_0)}{y} \right| dy < \infty.$$
Use Riemann-Lebesque Lemma to show that
$$\lim_{u\to\infty} S_u(x_0) = f(x_0).$$
(Hint: You can use $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$.)