NUMERICAL ANALYSIS QE: JUNE 30, 2016

Problem 1.(20 points) Let a be the root of the real-valued function $f(x)$ defined on the real line $\mathbb{R}$, that is, $f(a) = 0$. Suppose f is twice differentiable function on $\mathbb{R}$. Derive the Newton method for approximating the root a and derive the error estimate under a suitable condition of f.

Problem 2.(20 points) Let $\Omega = (0,1) \times (0,1)$ be the rectangle domain in the plane $\mathbb{R}^2$. Consider the boundary value problem of 2nd order for a function $u : \Omega \longrightarrow \mathbb{R}$:

$$\begin{aligned} -\Delta u + u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{1}$$

where Γ is the boundary of Ω and f is a given function on Ω. Let u_{ij} be the approximation of the function u at the nodal point (x_i, y_j) for $1 \leq i, j \leq N$ where N is an integer. Formulate the problem (1) into a matrix problem by using the finite difference method and discuss about the solvability of the resulting matrix problem.

Problem 3.(30 points) Let $f \in C^6[-1,1]$ and let $P \in \Pi_5$ be the Hermite interpolation polynomial with $P(x_i) = f(x_i), P'(x_i) = f'(x_i)$, $x_i = -1, 0, 1$.
(i) Show that

$$\int_{-1}^{1} P(t)dt = \frac{1}{15}(7f(-1) + 16f(0) + 7f(1)) + \frac{1}{15}(f'(-1) - f'(1)). \tag{2}$$

(ii) Derive the error formula for the integration rule (2). (iii) Write down the Peano kernel theorem. Show that the Peano kernel for (2) does not change its sign in $[-1,1]$.

Problem 4.(30 points) Let A be the matrix defined by

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 1 \end{bmatrix}. \tag{3}$$

Consider the Householder transformation $P = I - 2ww^H$ with $w^H w = 1, w \in \mathbb{C}^n$. Write down the properties of P. Using this P, find the QR-decomposition of the matrix A in (3), that is, $A = QR$ where Q is a unitary matrix and R is an upper triangular.