PhD QE: Manifold and Lie Groups

Dept. of Math. POSTECH

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There are 5 sets of problems altogether. Need to score 60 out of 100 points or better to pass. Answers with no proof will not count.

1. (20pts) Consider the two dimensional distribution Δ in $\mathbb{R}^3$ such that Δ_p, for every $p = (a, b, c) \in \mathbb{R}^3$, is given by

$$\Delta_p = \left\{ r \frac{\partial}{\partial x}\Big|_p + s \frac{\partial}{\partial y}\Big|_p + (rf(a,b) + sg(a,b)) \frac{\partial}{\partial z}\Big|_p : r, s \in \mathbb{R} \right\},$$

where $f, g : \mathbb{R}^2 \to \mathbb{R}$ are smooth functions depending only on x, y.

   (a) Find the integrability condition for the distribution Δ.

   (b) Find an equation for the integral manifold N with $(0, 0, 1) \in N$ when
   $$f = e^x \cos y, \qquad g = -e^x \sin y.$$

2. (20pts) Denote by $[\omega]$ the de Rham cohomology class of closed differential form ω on a smooth manifold. Let α and β be closed differential forms. Show that $\alpha \wedge \beta$ is also closed and the de Rham cohomology class of $\alpha \wedge \beta$ depends only on the de Rham cohomology classes of α and β — that is $\alpha \wedge \beta$ and $\alpha' \wedge \beta'$ have the same cohomology class whenever $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$.

3. (20pts) Let M be a connected smooth manifold and $\gamma : [0, 1] \to M$ be a smooth path on M. Let $\pi : \tilde{M} \to M$ be a smooth universal covering of M and ω be a closed differential 1-form on M. Then, by the path lifting property, there is a smooth path $u : [0, 1] \to \tilde{M}$ such that $\pi \circ u = \gamma$.

(a) Show that there is a smooth function $f : \tilde{M} \to \mathbb{R}$ such that

$$\int_\gamma \omega := \int_{[0,1]} \gamma^* \omega = f(u(1)) - f(u(0))$$

(b) Let $v : [0,1] \to \tilde{M}$ be another lifting of the path $\gamma : [0,1] \to M$. Show that

$$f(u(1)) - f(u(0)) = f(v(1)) - f(v(0))$$

4. (20pts) Consider the differential 2-form

$$\omega = \frac{(xdy \wedge dz - ydx \wedge dz + zdx \wedge dy)}{(x^2 + y^2 + z^2)^{3/2}}$$

defined on $U = \mathbf{R^3} - \{\mathbf{0}\}$.

(a) Show that $d\omega = 0$

(b) Let $S \subset U$ be an oriented closed smooth surface. Then, what are the possible values of $\int_S \omega$ and why?

5. (20pts) Let G be a connected Lie group and $\mathfrak{g} = T_e G$ the associated Lie algebra. Then we have a canonical trivialization

$$\omega_G : TG \to T_e G = \mathfrak{g}$$

defined by, $\forall v \in T_g G$,

$$\omega_G(v) = L_{g^{-1}*}(v),$$

where $L_h : G \to G$ is the left translation $L_h(g) = hg$. Regarding ω_G as a Lie algebra $\mathfrak{g}$-valued 1-form on G, show:

(a) ω_G is left-invariant.

(b) $\omega_G(X)$ is constant for any left-invariant vector field X.

(c) ω_G satisfies the Maurer-Cartan equation: $d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0$.