Complex Analysis Qualifying Exam
Summer 2016

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(1) (16 points) Suppose that f is an entire function defined everywhere in $\mathbb{C}$ such that for any $z_0 \in \mathbb{C}$, at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is equal to 0. Prove that f is a polynomial.

(2) (18 points) Show that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{\cosh \pi x} dx = \frac{1}{\cosh \pi \xi}$$

where $\cosh z = \frac{e^z + e^{-z}}{2}$.

(3) (17 points) Suppose that Q is a polynomial of degree $n \geq 2$ with distinct roots, none lying on the real axis. Calculate

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{Q(x)}\, dx \quad \text{for} \quad \xi > 0$$

in terms of the roots of Q.

(4) (17 points) Show that if f is holomorphic in the unit disc $\overline{D_1(0)} = \{z \in \mathbb{C} : |z| \leq 1\}$, is bounded, and converges uniformly to zero in the sector $\frac{\pi}{4} < \arg z < \frac{\pi}{3}$ as $|z|$ tends to 1, then f is identically 0 in the unit disc.

(5) (17 points) Let f be non-constant and holomorphic in an open set containing the closed unit disk $\overline{D_1(0)}$. Show that if $|f(z)| = 1$ on $C_1(0) = \{|z| = 1\}$, then the image of f contains the open unit disk $D_1(0) = \{z \in \mathbb{C} : |z| < 1\}$.
(Hint. Show that f has a zero in $D_1(0)$.)

(6) (15 points) Let Ω be open and connected. Suppose that a sequence

$$\{f_n : \Omega \to \mathbb{C} : f_n \text{ is holomorphic and injective in } \Omega\}$$

converges uniformly to a function f on every compact subset of Ω. Prove that if f is not a constant in Ω, then f is injective in Ω.