Real Analysis Ph.D. Qualifying Examination. January 2016, Postech

4 Problems. Justify all your work suitably. *Notation.* Let $dx = dm$ (or dy) denote Lebesgue measure in $\mathbb{R}^d$, and let $|E|$ be Lebesgue measure of a set E in $\mathbb{R}^d$.

1. (25 points) Let $T(x)$ be a Lipschitz transformation of $\mathbb{R}^2$ into $\mathbb{R}^2$. That is, there is a constant $C < \infty$ such that $|T(x) - T(y)| \leq C|x - y|$.

(a) Show that if E is a set in $\mathbb{R}^2$ with (Lebesgue) measure zero, then $T(E)$ has measure zero in $\mathbb{R}^2$.

(b) Show that if A is a measurable set in $\mathbb{R}^2$, then $T(A)$ is also a measurable set in $\mathbb{R}^2$. (Suggestion. Use the properties of compact sets.)

2. (25 points) Given $f \in L^1(\mathbb{R})$, define $G(x) = \int_{(-\infty,x]} f(t)\,dt$. Show that G is absolutely continuous on $\mathbb{R}$. (In particular, G is uniformly continuous on $\mathbb{R}$.)

3. (25 points) Given $f,\, g \in L^1(\mathbb{R}^d)$, define the convolution $f * g$ on $\mathbb{R}^d$ by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - y)\, g(y)\, dy.$$

If g is also bounded, show that

$$\lim_{|x| \to \infty} (f * g)(x) = 0.$$

4. (25 points) (a) Suppose that f_n converges in measure to f on $\mathbb{R}^d$. (Namely, for every $\varepsilon > 0$, $|\{x \in \mathbb{R}^d : \; |f_n(x) - f(x)| > \varepsilon\}| \to 0$ as $n \to \infty$.) Show that f_n has a subsequence f_{n_k} which converges to f a.e. in $\mathbb{R}^d$.

(b) Suppose that $\lim_{n\to\infty} \|f_n - f\|_{L^p(\mathbb{R}^d)} = 0$ for some $1 \leq p < \infty$. Show that f_n has a subsequence f_{n_k} which converges to f a.e. in $\mathbb{R}^d$.