Qualifying Exam Numerical Analysis January, 2016

1. (20 points) Let $y_m = \int_0^1 \frac{x^m}{x+5} dx, \ m = 0, 1, 2, \cdots$.

   (a) Derive a recurrence relation relating y_{m-1} to y_m. Represent the exact value y_0 as the Taylor series at center $c = 1$.

   (b) Consider the problem Q to compute y_0 by the recursion with initial term y_n:

$$y_n \longrightarrow \boxed{Q} \longrightarrow y_0$$

Determine the condition number K_n of y_0 at y_n and find $\lim_{n\to\infty} K_n$. Discuss about the numerical significance of your answers when $n \to \infty$.

2. (20 points) Let $x_0, x_1, \ldots, x_n$ be distinct points in (a, b). Let $p_n(x)$ be the polynomial which interpolates the function $f(x) = \sin(5x)$. Prove that

$$\lim_{n\to\infty} p_n(x) = f(x) \qquad \text{for every } x \in (a, b).$$

3. (20 points)

   (a) Find the Legendre polynomials $L_0(x), L_1(x), L_2(x), L_3(x)$ and $L_4(x)$ over the interval $[-1, 1]$.

   (b) Consider the Gaussian Quadrature rule of four points over the interval [-1,1] such that

$$\int_{-1}^{1} f(x)dx \approx Q_4(f) := w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) + w_4 f(x_4).$$

   Find the numbers of the nodes x_1, x_2, x_3, x_4 and derive some formulae for the weights w_1, w_2, w_3, w_4.

   (c) Find the degree of exactness (precision) of Q_4 with your work.

   (d) Prove that all weights are positive; $w_i > 0$ for $i = 1, 2, 3, 4$.

4. (20 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}.$$

Find the matrices L, U, P such that $LU = PA$, where P is the permutation matrix which is obtained by the partial pivoting.

5. (20 points) Let $A_{1\times 3} = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}$.

   (a) Find the singular value decomposition (SVD) of A: $A = U\Sigma V^T$.

   (b) Find the pseudoinverse A^+ of A.

The End