1. [20] Given a probability space $(\Omega, \mathscr{F}, P)$, prove or disprove the following statements:
   (a) [10] For bounded random variables X and Y, $E[YE(X|\mathscr{G})] = E[XE(Y|\mathscr{G})]$ for any sub σ-field $\mathscr{G}$.
   (b) [10] For sub σ-fields $\mathscr{G}_1 \subset \mathscr{G}_2$ and a random variable X such that $EX^2 < \infty$,
   $$E[X - E(X|\mathscr{G}_1)]^2 \geq E[X - E(X|\mathscr{G}_2)]^2.$$

2. [20] Let $X_1, X_2, \ldots$ be independent, satisfying (i) $\sum_{i=1}^n P(|X_i| > b_n) \to 0$, and (ii) $b_n^{-2} \sum_{i=1}^n E(X_i^2 1(|X_i| \leq b_n)) \to 0$, for some positive $b_n \uparrow \infty$. Prove that
   $$\frac{S_n - a_n}{b_n} \to 0$$
   in probability where $S_n = \sum_{i=1}^n X_i$ and $a_n = \sum_{i=1}^n E(X_i 1(|X_i| \leq b_n))$.

3. [20] Let $X_1, X_2, \ldots$ be independent and identically distributed with probabilities
   $$P(X_1 = 2^k - 1) = \frac{1}{k(k+1)2^k}, \quad k = 1, 2, \ldots,$$
   and $P(X_1 = -1) = 1 - \sum_{k=1}^\infty P(X_1 = 2^k - 1)$. Let
   (a) [5] Show that $E(X_1) = 0$.
   (b) [15] Show that for $S_n = \sum_{i=1}^n X_i$
   $$P\left(S_n < -\frac{n}{2\log_2 n}\right) \to 1.$$

4. [20] Let $X_1, X_2, \ldots$ be a sequence of random variables such that
   (i) $X_{n+m} - X_n$ is independent of $\mathscr{F}(X_1, \ldots, X_n)$,
   (ii) the distribution of $X_{n+m} - X_n$ does not depend on n
   for every $n, m = 1, 2, \ldots$. Verify that the statements (i) and (ii) hold even if n is replaced by a stopping time n^*.

5. [20] Let W_t, $t \geq 0$, be a standard Brownian motion. Consider two parallel lines $y = a + rt$ and $y = -b + rt$, for positive a, b, r, and let T be the first time W_t meets either line, and $p(a, b, r)$ be the probability that the exit is through the upper barrier, i.e., $W_T = a + rT$.
   (a) [10] Show that $Y_t(\theta) = \exp(\theta W_t - \frac{1}{2}\theta^2 t)$ is a martingale, and deduce that $E[Y_t(\theta)] = 1$ for all θ. Argue that for all θ, $E\left[\exp(\theta W_T - \frac{1}{2}\theta^2 T)\right] = 1$.
   (b) [10] Obtain $p(a, b, r)$. What is the limiting value of $p(a, b, r)$ as $r \to 0$?