Qualifying Exam: Algebraic Topology 2015

1. (15 points) Let X be a topological space. Define the suspension $S(X)$ to be the space obtained from $X \times [0,1]$ by contracting $X \times \{0\}$ to a point and contracting $X \times \{1\}$ to another point. Describe the relation between the homology groups of X and $S(X)$.

2. (15 points)

   Let X be the topological space identified from a hexagon by identifying its edges cyclically. Calculate the homology and cohomology groups of X.

3. (15 points) Suppose that X is a CW complex and Y is obtained from X by attaching cells of dimension $> n$. Show that $H_i(Y) = H_i(X)$ for $i < n$ and $H_n(Y)$ is a quotient of $H_n(X)$.

4. (15 points) Determine $H_k(X)$ for each k where

$$X = \{(x, y, z) \in \mathbb{R}^3 \; : \; (x^2 + y^2 - 1)(x^2 + z^2 - 9) = 0\}.$$

5. (15 points)

   Let $X := \mathbb{R}P^2 \times \mathbb{R}P^4 \times \mathbb{R}P^6 \times \mathbb{R}P^8$. Show that any continuous map from X to itself has a fixed point. (hint: You can use the Lefschetz fixed point theorem; if X is a finite simplicial complex, and $f : X \to X$ is a continuous map with $\tau(f) := \sum_n (-1)^n \operatorname{tr}(f_* : H_n(X, \mathbb{Q}) \to H_n(X, \mathbb{Q})) \neq 0$, then f has a fixed point.)

6. (10 points)

   (a) Calculate $\operatorname{Tor}_1^{\mathbb{Z}/30\mathbb{Z}}(\mathbb{Z}/10\mathbb{Z}, \mathbb{Z}/5\mathbb{Z})$ in the category of $\mathbb{Z}/30\mathbb{Z}$-modules.

   (b) Calculate $\operatorname{Ext}^1_{\mathbb{Z}/16\mathbb{Z}}(\mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$ in the category of $\mathbb{Z}/16\mathbb{Z}$-modules

7. (15 points)

   Show that there is no compact three-dimensional manifold M whose boundary is the real projective space $\mathbb{R}P^2$.