PhD QE: Manifold and Lie Groups

Dept. of Math. POSTECH

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There are 5 sets of problem altogether. Need to score 60 out of 100 points or better to pass. A correct answer with no proof counts for nothing.

1. (15pts) Let $E_t = \{(x, y) \in \mathbf{R}^2 | y^2 = x(x-1)(x-t)\}$, where t is a real parameter. Find every value of t that E_t is a smooth manifold.

2. (15pts) Consider the differential 1-form

$$\omega = \frac{(-ydx + xdy)}{x^2 + y^2}$$

defined on $U = \mathbf{R^2} - \{\mathbf{0}\}$. Let γ be a piecewise smooth simple-closed curve on U. Then, what are the possible values of $\int_\gamma \omega$ and why?

3. (30pts) Let M be a smooth manifold, $\mathcal{X}(M)$ be the space of smooth vector fields on M and $\Lambda^k(M)$ be the space of differential k-forms on M.

The Lie derivative L_X with respect to $X \in \mathcal{X}(M)$ is a linear operator $L_X : \Lambda^k(M) \to \Lambda^k(M)$ defined by

$$(L_X\omega)(X_1, \ldots, X_k) = X(\omega(X_1, \ldots, X_k)) - \sum_{i=1}^{k} \omega(X_1, \ldots, X_{i-1}, [X, X_i], X_{i+1}, \ldots X_k),$$

where $X_1, \ldots, X_k \in \mathcal{X}(M)$ and $[\ ,\ ]$ denote the Lie bracket on $\mathcal{X}(M)$.

The interior differential operator ι_X with respect to $X \in \mathcal{X}(M)$ is a linear map $\iota_X : \Lambda^k(M) \to \Lambda^{k-1}(M)$ defined by

$$(\iota_X\omega)(X_1, \ldots, X_{k-1}) = k\omega(X, X_1, \ldots, X_{k-1})$$

(a) Show that $L_X\iota_Y - \iota_Y L_X = \iota_{[X,Y]}$.

(b) Show that $L_X = d\iota_X + \iota_X d$.

(c) Show that $L_X L_Y - L_Y L_X = L_{[X,Y]}$.

4. (10pts) A smooth manifold M is called a symplectic manifold if there a symplectic form $\omega \in \Lambda^2(M)$, which is a closed nondegenerate differential 2-form – there is an isomorphism $\tilde{\omega} : \mathcal{X}(M) \to \Lambda^1(M)$ defined by $\tilde{\omega}(X)(Y) := \omega(X, Y)$.

   Let (M, ω) be a symplectic manifold. Given $f, g \in \Lambda^0(M) = C^\infty(M)$, the Poisson bracket of f and g is defined by

$$\{f, g\} = X_f g = \omega(X_g, X_f)$$

   where $X_f := \tilde{\omega}^{-1}(df) \in \mathcal{X}(M)$ or, equivalently, $\iota_{X_f}\omega = df$.

   Show that $(C^\infty(M), \{\ ,\ \})$ is a Lie-algebra.

5. (30pts) Let $H \subset G$ be a connected Lie subgroup of a connected Lie group G. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. Show that the homogeneous manifold G/H is a Lie group if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. (Do not prove the existence of differentiable manifold structure in G/H).