Complex Analysis QE

1. Let $\Omega \subset \mathbb{C}$ be an open set. Let γ be a closed piecewise-smooth curve in Ω. Suppose that f is holomorphic in Ω. Define g on $\Omega \times \Omega$ by

$$g(z,w) := \begin{cases} \dfrac{f(w)-f(z)}{w-z} & \text{if } w \neq z \\ f'(z) & \text{if } w = z \end{cases}$$

1.1. (10 points) Show that g is continuous in $\Omega \times \Omega$.

1.2. (15 points) Define h in Ω by

$$h(z) := \frac{1}{2\pi i}\int_\gamma g(z,w)dw.$$

Show that h is holomorphic and $h = 0$.

2. Suppose

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n, \qquad |z-a| < R.$$

2.1. (10 points) Show that for $0 < r < R$,

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(a+re^{i\theta})\right|^2 d\theta.$$

2.2. (15 points) Show that if $R > 1$, then

$$\sum_{n=0}^{\infty} |c_n| < \infty.$$

3. (15 points) Evaluate the integral

$$\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}dx, \quad a > b > 0.$$

4. Let f be a non-constant holomorphic function in a bounded open connected set Ω.

4.1. (10 points) Suppose that $D := \{z: |z-a| < r\} \subset \Omega$. Show that there exists $z_1 \in D$ such that $|f(z_1)| > |f(a)|$.

4.2. (10 points) Let $M := \limsup_{n\to\infty} |f(z_n)|$ for every sequence $\{z_n\}$ in Ω which converges to a boundary point of Ω. Show that $|f(z)| < M$ for $z \in \Omega$.

5. (15 points) Show that if f is holomorphic and nonconstant in an open connected domain Ω, then $f(\Omega)$ is an open set.