2015년 대수학II 박사자격시험(2015 Fall)

Show all the steps. Otherwise you may not get points.

1. Prove that ut C is an uncountable set,
   where Aut C is the set of field automorphisms of C.

2. Let F be the splitting field of Q over $f(x) = x^5 - 5$.
   Find the number of subfields of F.

3. Let F≥K be a field extension.
   Prove that if u ∈ F is purely inseparable over K and v ∈ F is purely inseparable over K, then K(u,v)=K(u+v).

4. Let F be an algebraic closure of a finite field K.
   (a) Prove that $Aut_K F$ is an abelian group.
   (b) Prove that F does not have a subfield with finite index.

5. Let K be a finite extension field of a field F with characteristic zero.
   Show that there exists an element u ∈ K such that K=F(u).

6. Suppose that f(x) is an irreducible polynomial of order 5 over Q, which has exactly 3 real roots in C.
   (a) Prove that the Galois group of a splitting field of f(x) over Q is the symmetric group S_5.
   (b) Is f(x) solvable by radicals?

7. Let K be a Galois extension of F with finite cyclic Galois group G generated by σ.
Prove that $\alpha \in$ K has the norm $N_{K/F}(\alpha)=1$
if and only if α is of the form $\alpha=\frac{\beta}{\sigma(\beta)}$ for some nonzero $\beta \in$ K.

8. Prove that $Q(\sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}) = Q(\sqrt{2} + \sqrt{3} + \ldots + \sqrt{n})$ for all natural number n≥2.

9. Let $p_1, p_2, \ldots, p_n$ be prime numbers.
Is it true that $Q(\sqrt[3]{p_1}, \sqrt[3]{p_2}, \ldots, \sqrt[3]{p_n}) = Q(\sqrt[3]{p_1} + \sqrt[3]{p_2} + \ldots + \sqrt[3]{p_n})$, where $\sqrt[3]{p_i}$ denote the positive cube root of p_i ?
Verify your answer.