Qualifying Exam: Real Analysis
Summer, 2015

6 problems, 100 points

1.(15 points) Suppose $\{\alpha_j\}_{j=1}^{\infty} \subset (0,1)$. Show that $\Pi_{j=1}^{\infty}(1-\alpha_j) > 0$ if and only if $\sum_{j=1}^{\infty} \alpha_j < \infty$.
(Hint. Compare $\sum_{j=1}^{\infty} \log(1-\alpha_j)$ to $\sum_{j=1}^{\infty} \alpha_j$.)

2.(15 points) Let (X, μ) be a measure space with a (positive) measure μ.
Let $\{f_n : X \to \mathbb{R} : n = 1, 2, \cdots\}$ be a sequence of measurable functions which satisfy the following properties:

(i) For each $n = 1, 2, \cdots$, f_n satisfies $|f_n| \leq g$ where $g \in L^1(X, \mu)$,

(ii) f_n converges to a measurable function f almost everywhere on X.

Prove the following statement: *For any given constant $\varepsilon > 0$, there exists a measurable set $E \subset X$ such that $\mu(E) < \varepsilon$, and f_n converges to f uniformly on $X \setminus E$.*

(CAUTION: This is a variation of Egoroff's Theorem. DO NOT simply quote Egoroff's theorem to prove the statement.)

3.(25 points) In $\mathbb{R}^n$, set $B_r(x) := \{y \in \mathbb{R}^n : |y - x| < r\}$ for each $r > 0$. And, let $m(E)$ be the Lebesgue measure of a Lebesgue measurable set $E \subset \mathbb{R}^n$

For $f \in L^1_{loc}(\mathbb{R}^n)$, *Hardy-Littlewood maximal function* Hf is defined by

$$Hf(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y)|\, dy.$$

(a) (15 points) Prove *the Maximal Theorem*. In other words, show that there exists a constant $C > 0$ such that for any $f \in L^1(\mathbb{R}^n)$ and for any $\alpha > 0$, Hf satisfies

$$m(\{x \in \mathbb{R}^n : Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}$$

(b) (10 points) Suppose that $f \in L^1(\mathbb{R}^n)$ and $\|f\|_{L^1(\mathbb{R}^n)} > 0$. Show that there exist constants $C, R > 0$ such that

$$Hf(x) \geq C|x|^{-n} \quad \text{for} \quad |x| > R.$$

4.(15 points) For a Banach space $\mathcal{B}$, set $\partial B_1(\mathbf{0}) := \{w \in \mathcal{B} : \|w\|_{\mathcal{B}} = 1\}$. For a bounded linear mapping $T : \mathcal{B} \to \mathcal{B}$, the norm $\|T\|$ of the mapping T is defined by

$$\|T\| := \sup_{w \in \partial B_1(\mathbf{0})} \|Tw\|_{\mathcal{B}}.$$

Prove the following statement: If a bounded linear mapping $T : \mathcal{B} \to \mathcal{B}$ satisfies

$$\|I - T\| < 1$$

where I is the identity mapping (i.e., $Iw = w$ for all $w \in \mathcal{B}$), then T is invertible.

5.(15 points) For a Hilbert space $\mathcal{H}$ equipped with an inner product $\langle \cdot, \cdot \rangle$, define the norm of $w \in \mathcal{H}$ by

$$\|w\| = \sqrt{\langle w, w \rangle}.$$

Fix $u \in \mathcal{H}$, and a constant $R_0 > 0$. Define a subset $K_u(R) \subset \mathcal{H}$ by

$$K_u(R) = \{w \in \mathcal{H} : \langle w, u \rangle \geq R_0\}.$$

Show that there exists a unique $w_0 \in K_u(R)$ such that

$$\|w_0\| = \inf_{w \in K_u(R)} \|w\|.$$

Hint. Use the Parallelogram Law: For all $p, q \in \mathcal{H}$,

$$\|p + q\|^2 + \|p - q\|^2 = 2(\|p\|^2 + \|q\|^2).$$

6.(15 points) Let (X, μ) be a measure space with a (positive) measure μ. Suppose that $f \in L^p(X, \mu) \cap L^\infty(X, \mu)$ for some $1 < p < \infty$. Show that

$$\lim_{q \to \infty} \|f\|_{L^q(X)} = \|f\|_{L^\infty(X)}.$$