NUMERICAL ANALYSIS QE: JULY, 2015

Problem 1.(20 points) Consider the following boundary value problem of 2nd order for a function $u : [0,1] \longrightarrow \mathbb{R}$:

$$\begin{aligned} -u''(x) + q(x)u(x) &= f(x), \quad x \in (0,1), \\ u(0) = 0, \quad u(1) &= 0. \end{aligned} \tag{1}$$

where f and q are given functions on $[0,1]$.

(i) Formulate the problem (1) into a matrix problem by using the finite difference method. Hint. Consider $x_0 < x_1 < \cdots < x_{n+1} = 1$ and use the notation u_i as the approximation for the exact value $u(x_i)$. (ii) Discuss the error estimate for the approximation u_i.

Problem 2.(20 points) (i) Write down the Peano kernel theorem. (ii) Let $f(x)$ be a four times differentiable real-valued function on the interval $[-1,1]$. Derive the following equality

$$\int_a^b f(x)dx = \frac{1}{2}(f(a)+f(b)) + \frac{h^2}{12}[f'(a) - f'(b)] + \frac{b-a}{720}h^4 f^{(4)}(\xi),\ \xi \in (a,b),$$

where $h = b - a$. (Hint: Use the Peano kernel).

Problem 3.(20 points) Let $p_0, p_1, \cdots, p_n$ be the orthogonal polynomials by the Gram-Schmidt orthogonalization. Let $x_1, \cdots, x_n$ be the roots of the orthogonal polynomial $p_n(x)$ of degree n. Suppose that $w_1, \cdots, w_n$ are the solution of the system of equations

$$\begin{aligned} \sum_{i=1}^n p_k(x_i)w_i &= (p_0, p_0), \quad (k = 0), \\ \sum_{i=1}^n p_k(x_i)w_i &= 0, \quad (k = 1, \cdots, n-1), \end{aligned} \tag{2}$$

Show that $\omega_i > 0$ for $i = 1, \cdots, n$ and

$$\int_a^b p(x)\omega(x)dx = \sum_{i=1}^n w_i p(x_i), \qquad \forall p \in \Pi_{2n-1}$$

where $\omega(x)$ is a weight function and Π_{2n-1} is the set of all polynomials with degrees less than or equal to $2n-1$.

Problem 4.(30 points) Let $f \in C^6[-1,1]$ and let $P \in \Pi_5$ be the Hermite interpolation polynomial with $P(x_i) = f(x_i), P'(x_i) = f'(x_i)$, $x_i = -1, 0, 1$.
(i) Show that

$$\int_{-1}^{1} P(t)dt = \frac{1}{15}(7f(-1) + 16f(0) + 7f(1)) + \frac{1}{15}(f'(-1) - f'(1)). \tag{3}$$

(ii) Derive the error formula for the integration rule (3). (iii) Show that the Peano kernel for (3) does not change its sign in $[-1,1]$.

Problem 5.(10 points) Let A be the matrix defined by

$$A = \begin{bmatrix} 0 & 8 & 3 \\ 0 & 6 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4}$$

Give the definition of the singular value decomposition for any $m \times n$ matrix A. Find the singular value decomposition for the matrix A of (4). Find the pseudoinverse A^+ of A.