2015 PhD QE (Spring) **Probability & Statistics**

1. [20] Let X be a random variable with mean 0 and variance 1. Prove that

$$P(X \geq c) \leq \frac{1}{1+c^2}, \quad c \geq 0,$$

and indicate when the equality holds.

2. [20] Let $\{X_n\}$ be random variables with $EX_i = 0$ and $EX_iX_j \leq r(j-i)$, $1 \leq i \leq j < \infty$, where $r(n)$ is a sequence of real numbers converging to 0 as $n \to \infty$. Show that $n^{-1}\sum_{i=1}^n X_i \to 0$ in probability.

3. (a) [20] Let $X_1, X_2, \dots$ be independent, satisfying
   (i) $\sum_{i=1}^n P(|X_i| > b_n) \to 0$,
   (ii) $b_n^{-2}\sum_{i=1}^n E(X_i^2 1(|X_i| \leq b_n)) \to 0$,
   for some positive $b_n \uparrow \infty$. Prove that

$$\frac{S_n - a_n}{b_n} \to 0$$

   in probability where $S_n = \sum_{i=1}^n X_i$ and $a_n = \sum_{i=1}^n E(X_i 1(|X_i| \leq b_n))$.

   (b) [20] Let $X_1, X_2, \dots$ be independent and identically distributed with probabilities

$$P(X_1 = 2^k - 1) = \frac{1}{k(k+1)2^k}, \quad k = 1, 2, \dots,$$

   and $P(X_1 = -1) = 1 - \sum_{k=1}^\infty P(X_1 = k)$. Let
   (i) Show that $E(X_1) = 0$.
   (ii) Show that for $S_n = \sum_{i=1}^n X_i$

$$P\left(S_n < -\frac{n}{2\log_2 n}\right) \to 1.$$

4. [20] Let $\{X_n; n = 0, 1, \dots\}$ be a martingale, and let τ be stopping time such that $0 \leq \tau \leq K$ with probability 1 for some integer K. Verify that $E(X_\tau) = E(X_0)$.