# Differential Manifolds and Lie Groups Qualifying Exam

Please note that the total is forty-two points, and grading is done at the increment of one fifth of a point. The minimum passing mark is twenty-five and a fifth (corresponds to sixty percent).

1. (a) Show that a compact differentiable manifold can always be embedded in $\mathbb{R}^n$ for $n$ sufficiently large.

   [7 points]

   (b) Show that the subset

   $$\{g \in GL_n(\mathbb{R}) : g^t g = I\}$$

   is a compact Lie group. (Here $I$ means the identity matrix.)

   [7 points]

2. (a) Let $\alpha$ be a 1-form on a smooth manifold, and let $\mathcal{D} = ker(\alpha)$. Show that $\mathcal{D}$ is involutive if and only if $\alpha \wedge d\alpha = 0$.

   [5 points]

   (b) Let $M$ be a smooth manifold of dimension 3 and $\beta$ a smooth 1-form such that $\beta \wedge (d\beta) \neq 0$ everywhere on $M$. Show that the distribution given by the kernel of $\beta$ is not integrable.

   [3 points]

   (c) In $\mathbb{R}^3$ let $\alpha = dz - xdy$ and let $\mathcal{D}$ be distribution given by the kernel of $\alpha$. Determine whether $\mathcal{D}$ is integrable or not.

   [2 points]

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

   $$f(x, y) = x^3 - 6xy + y^2.$$

   Find all values $c \in \mathbb{R}$ such that $f^{-1}(c)$ is a regular submanifold of $\mathbb{R}^2$.

   [6 points]

4. Give an immersion that is not an embedding. (You need to include in your answer a justification that it is an immersion and why it is not an embedding.)

   [6 points]

5. Show that on a simply connected smooth manifold, all closed 1-forms are exact.

   [6 points]