Qualifying Examination - Complex Analysis

July 2015

Let $D_r = \{z \in \mathbb{C} : \ |z| < r\}$ for $r > 0$, and let $D = D_1$.

1. (17 points) Let $f = u + iv$ be a holomorphic function in D such that v is nonnegative and $v(0) = 0$. Determine all such functions f.

2. (16 points) Let f, h be holomorphic functions in an open set G containing $\overline{D_2}$. Assume that f has zeros at $z_1, z_2, \cdots, z_k$ in D_2 and no other zeros in G. Let C be the circle $|z| = 2$, traversed once counterclockwise. Evaluate $\oint_C \frac{f'(z)}{f(z)} h(z)\, dz$.

3. (17 = 13+4 points) Suppose that $f(z)$ is holomorphic in an open set G containing $\overline{D_2}$ and that $|f(z)| < 4$ on the circle $|z| = 2$. (i) Show that the equation $f(z) = z^2$ has exactly two solutions a, b in D_2. (ii) Give an example of f such that $a \neq b$.

4. (17 points) Suppose that $f : D \to \mathbb{C}$ is a nonconstant holomorphic function such that $|f(z)| \leq 1$ for all $z \in D$. Explain why $|f(z)| < 1$ for all $z \in D$, and then show that for every $z \in D$,

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}.$$

5. (16 points) Let u_k be a sequence of harmonic functions in an open set $G \subset \mathbb{C}$. Show that if u_k converges uniformly on every compact subset of G, then the limit function u is also harmonic in G.

6. (17 points) Use Hadamard's theorem to show that

$$e^z - 1 = e^{z/2} \cdot z \cdot \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$