QUALIFYING EXAMINATION IN ALGEBRA
JULY, 2015

(1) Let $V = \mathbb{R}^n$ be a vector space of dimension n over $\mathbb{R}$ and let

$$0 = V_n \subset V_{n-1} \subset \cdots V_1 \subset V = V_0$$

be a decresing sequence of subspaces of the vector space V such that $\text{codim}(V_i) = i$ for each i (so that $V_1 \cong \mathbb{R}^{n-1}, V_2 \cong \mathbb{R}^{n-2}, \cdots$ so on).

Let G be a group of invertible matrices such that

$$G = \{g \in GL(V) \mid gV_i = V_i, 0 \leq i \leq n\}.$$

Definie a decresing sequence of subgroups $(B_i)_{0 \leq i \leq n}$ of G by

$$B_i = \{g \in G \mid (g - I)V_j \subset V_{i+j}, 0 \leq j \leq n - i\}.$$

(a) (15pts) Let (B_j, B_k) be a group generated by commutators of the form $x^{-1}y^{-1}xy$ with $x \in B_j, y \in B_k, 0 \leq i, k \leq n$. Show that

$$(B_j, B_k) \subset B_{j+k}, \text{ if } 0 \leq j + k \leq n.$$

(b) (5pts) Show that $B_i, 0 \leq i \leq n$, is a normal subgroup of $B_0 = G$.

(c) (5pts) Show that B_i/B_{i+1} are abelian for $1 \leq i \leq n - 1$ and $B_0/B_1 = G/B_1$ is also abelian.

(d) (5pts) Conclude that G is solvable.

(2) Show the following:

(a) (7pts) $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$ with $d = \gcd(m, n)$.

(b) (6pts) $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

(c) (7pts) Is $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ an integral domain? Justify your answer.

(3) (10pts) Let P be a group of order $p^a, a \geq 1$, for prime p. Show that the center $C(P)$ of P is nontrivial.

(4) For any integral domain D let $D^* = D^\times \cup \{0\}$ denote the collection of units of D together with 0. An element $u \in D - D^*$ is called a **universal side divisor** if for every $x \in D$ there is some $z \in D^*$ such that u divides $x - z$ in D.
Now consider $R = \mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ a ring of quadratic integers.
(a) (20pts) Show that R doesnt have a universal side divisor.

(b) (20pts) Show that all of the ideals of R are principal.

THE END