Real Analysis Qualifying Examination

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(Justify all your work. You may use the results of problem A to solve problem B.)

1.1. (15 points) Let μ^* be a metric outer measure on subsets in R^n : If $\mathrm{dist}(E_1, E_2) > 0$, then $\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$. Let B be a Borel set. Show that

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A - B)$$

for any $A \subset R^n$.

1.2. (15 points) Let μ^* be the Lebesgue outer measure. Prove that there exist disjoint bounded subsets E, F of the real line such that

$$\mu^*(E \cup F) \neq \mu^*(E) + \mu^*(F).$$

1.3. (15 points) Denote $A\Delta B := (A\backslash B) \cup (B\backslash A)$. Let E be a set in R^n. Suppose that for any $\epsilon > 0$, there exists a Lebesgue measurable set F such that $\mu^*(E\Delta F) < \epsilon$. Show that E is Lebesgue measurable. (Suggestion: Find a measurable set $\tilde{E} \supset E$, $\mu^*(\tilde{E}\Delta E) = 0$.)

2. (15 points) Let $f \in L^1(R^n)$. Let $\epsilon > 0$ be given. Show that there exists a Lebesgue measurable set A with $\mu(A) < \epsilon$ such that the restriction of f to $R^n\backslash A$ is continuous. (Suggestion: Consider Tchebyshev's inequality: $\mu(\{x: |f(x)| > \alpha\}) \leq \|f\|/\alpha$.)

3. (15 points) Let $f \in L^1(\mathbb{R}^n)$ and define

$$f^*(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f|\, d\mu$$

where B denotes a ball.

Show that f^* is lower semi-continuous, $f^*(x) < \infty$ a.e. and $f^*(x) \geq |f(x)|$ a.e.

4. (15 points) For a fixed $a \in R^n$, define

$$f_{a,r} := \frac{1}{\mu(B(a,r))} \int_{B(a,r)} f(x)\, d\mu$$

Suppose $|\nabla f| \in L^p(R^n)$, $1 \leq p < n$, $s > n - p$, and

$$\limsup_{r \to 0} \frac{1}{r^s} \int_{B(a,r)} |\nabla f|^p\, d\mu < \infty.$$

Show that $f_{a,r} \nrightarrow \infty$ as $r \to 0$. (Suggestion: Consider $\{f_{a,2^{-k}} : k = 1,2,\cdots\}$.)

5. (10 points) Let $f\colon [a,b] \to \mathbb{R}$ satisfy

$$|f(x)-f(y)| \leq c|x-y|, \quad x,y \in [a,b].$$

so that f is differentiable a.e. Show by applying the dominated convergence theorem, (and don't cite the absolute continuity results), that

$$\int_a^b f'\, d\mu = f(b) - f(a).$$