2014 PhD QE (Fall) **Probability & Statistics**

– Choose one (and only one) of Section P or Section S. –
(You are not allowed to cross-select problems.)

[Section P]

1. Let $A_1, A_2, \ldots$ be a sequence of events such that (i) A_i and A_j be independent whenever $|i-j| \geq \ell$ for some integer $\ell > 0$, and (ii) $\sum_n P(A_n) = \infty$. Prove that $P(\limsup A_n) = 1$.

2. Let $X_1, X_2, \ldots$ be a sequence of random variables converging to X almost surely. Furthermore, there is $M > 0$ such that $|X_n| \leq M$ for all n.
Prove that $\lim E|X_n - X| = 0$.

3. Let $X, X_1, X_2, \ldots$ be random variables. Verify that $X_n \to X$ in probability if and only if
$$E\left(\frac{|X_n - X|}{1 + |X_n - X|}\right) \to 0 \quad \text{as } n \to \infty.$$

4. Let X be a random variable in a probability space $(\Omega, \mathscr{F}, P)$ such that $E|X| < \infty$. Verify that, if $\mathscr{G} \subset \mathscr{H}$ are sub-sigma fields of $\mathscr{F}$, then $E(X|\mathscr{G}) = E(E(X|\mathscr{H})|\mathscr{G})$ a.s.

5. Let $X_1, X_2, \ldots$ be independent and equal to ± 1 with probability $\frac{1}{2}$, and $S_n = X_1 + \cdots + X_n$.
   (a) Define $\boldsymbol{n} = \{\text{first } n \text{ such that } S_n > 0\}$. Prove that $E\boldsymbol{n} = \infty$.
   (b) Let r, s be positive integers. Define $\boldsymbol{m} = \{\text{first } n \text{ such that } S_n = 0 \text{ or } -s\}$. Prove that $E\boldsymbol{m} < \infty$.
   (c) Calculate $P(S_{\boldsymbol{m}}) = s/(r+s)$.

6. Let $\{X_n\}$ be a stochastic process such that $E(X_0) = 0$ and
$$E(X_{n+1}|X_0, X_1, \ldots, X_n) = X_n + 1, \quad n = 0, 1, \ldots.$$
Let τ be a stopping time for $\{X_n\}$ with $E(\tau) = m < \infty$. Calculate $E(X_\tau)$.

[Section S]

1. Let $X_1, \ldots, X_n$ be a random sample from a distribution with p.d.f. $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$, $x \in \Re$. Show that $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ follows the same distribution as X_1.

2. Let $\boldsymbol{X} \sim f(\boldsymbol{x};\theta)$, $\theta \in \Theta$, and $T = t(\boldsymbol{X})$ be a sufficient statistic for θ. A statistic $\psi(\boldsymbol{X})$ is called a *test function* if $\psi(\boldsymbol{x}) \in [0,1]$. Verify that for every test function $\psi(X)$, there exists a test function $\phi(T)$ such that $E_\theta\phi(T) = E_\theta\psi(\boldsymbol{X})$ for all $\theta \in \Theta$.

3. (a) Let $X \sim N(0,\theta)$, $\theta > 0$. Is X a complete sufficient statistic? If not, find one.,

Now let $X_1, \ldots, X_n$ be a random sample from $N(0,\theta)$, $\theta > 0$.

(b) What is the Fisher information about θ contained in $\bar{X} = \sum_{i=1}^n X_i/n$? What about $T = \sum_{i=1}^n X_i^2$? (c.f. The p.d.f. of χ_m^2 is $g(v) = v^{(m-2)/2}e^{-v/2}/2^{m/2}\Gamma(m/2)$, $v > 0$.)

(c) Find the UMVU estimator of θ.

4. The sizes of fishes in a pond is thought to follow an exponential distribution with mean μ centimeters. To estimate μ, some fishes were netted at random and measured as $X_1, X_2, \ldots, X_n$. However, the net can catch only the fishes that are larger than 10 cm.

(a) Find the MLE for μ and show that it is unbiased.

(b) Construct a uniformly most powerful test of level α for $H_0 : \mu \leq \mu_0$ vs $H_1 : \mu > \mu_0$, using a chi-square distribution for reference. (c.f. $\mathcal{X}_2^2$ distribution is the same as exponential distribution with mean 2.)

5. Suppose that $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are independent random samples with p.d.f.

$$f(x;\lambda) = \frac{1}{\lambda}e^{-x/\lambda}, x > 0, \quad \text{and} \quad g(y;\mu) = \frac{1}{\mu}e^{-y/\mu}, y > 0,$$

respectively, for some $\lambda > 0$ and $\mu > 0$. Let

$$Z_i = \min(X_i, Y_i) \quad \text{and} \quad W_i = \begin{cases} 1, & \text{if } Z_i = X_i, \\ 0, & \text{if } Z_i = Y_i \end{cases} \quad \text{for } i = 1, \ldots, n.$$

(a) Find the joint distribution of Z_1 and W_1, and show that they are independent of each other.

(b) Given (Z_i, W_i), $i = 1, \ldots, n$, find the MLE of λ.

(c) Show that the MLE obtained in (b) is asymptotically normally distributed, and derive an asymptotic level α test for $H_0 : \lambda = \lambda_0$ vs $H_0 : \lambda \neq \lambda_0$.