2014–2 Qualifying Exam
Algebraic Topology

1. (20pts) Using homology groups, prove Brouwer's fixed point theorem: any map $f\colon D^n \to D^n$ has a fixed point.

2. (20pts) Suppose $n \geq 1$ is an integer. Let $A = \{z \in S^1 \subset \mathbb{C} \mid z^n = 1\}$. Compute the relative homology groups $H_*(S^1 \times S^1, A \times \{1\} \cup \{1\} \times S^1)$.

3. (20pts) Using an induction on n and Mayer-Vietoris, show that $H_k(S^n \times X) \cong H_k(X) \oplus H_{k-n}(X)$. (Do not use Künneth.)

4. (20pts) Suppose R is a ring with $\frac{1}{2}$, $g\colon X \to X$ is a homeomorphism of a CW complex X sending each cell onto another distinct cell, and $g^2 = \mathrm{id}_X$. Let $p\colon X \to Y := X/x \sim g(x)$ be the quotient map. Show that $p_*\colon H_i(X;R) \to H_i(Y;R)$ is surjective.

5. (20pts) For a nonsingular $n \times n$ matrix A over $\mathbb{R}$ ($n \geq 2$), define $f_A\colon S^{n-1} \to S^{n-1}$ by $f(x) = Ax/|Ax|$ for $x \in S^{n-1} \subset \mathbb{R}^n$. Show that $\deg f_A = \det A/|\det A|$, by investigating the effect of elementary column (or row) operations on A.