Ph. D. QE — Manifolds & Lie Groups

Dept. of Math., POSTECH

January 2015

Note: There are 6 problems altogether. The weight of each problem is specified. 60 points or better will pass. Incomplete answers will receive very little or no credits; there are essentially no partial credits.

1. (20 pts) Let M be a compact manifold of dimension n, and let $f\colon M \to \mathbb{R}^n$ be a smooth map. Show that there exists a point $p \in M$ such that its differential $df_p\colon T_pM \to T_{f(p)}\mathbb{R}^n$ fails to be a linear isomorphism.

2. (10 pts) For vector fields $X_0, \dots, X_k$ and a differential $(k+1)$-form ω on a differentiable manifold M, the interior differentiation operator $i^{k+1}_{X_0}\colon \Lambda^{k+1}(M) \to \Lambda^k(M)$ is defined by

$$(i^{k+1}_{X_0}\omega)(X_1, \dots, X_k) = \omega(X_0, X_1, \dots, X_k)$$

for every integer $k > 0$. Show that $i^k_{X_0} \circ i^{k+1}_{X_0} = 0$ for any positive integer k.

3. (15 pts) Let ω be a differential 1-form and let X, Y be smooth vector fields on a differentiable manifold M. Prove the following identity by É. Cartan:

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]).$$

4. (20 pts) Let α be a differential 1-form on a smooth manifold M that vanishes nowhere. Then

$$\ker \alpha := \{X \text{ smooth vector field}\colon \alpha(X) = 0\}$$

is a smooth distribution. (Do not prove this fact). Show that $\ker \alpha$ is integrable if and only if $\alpha \wedge d\alpha = 0$.

5. (15 pts) Consider the differential 1-form

$$\alpha = (x^2 + 7y)dx + (y \sin y^2 - x)dy$$

defined on $\mathbb{R}^2$. Suppose that γ is a simple-closed piecewise smooth curve in $\mathbb{R}^2$, then what can you say about the value of $\int_\gamma \alpha$?

6. (20 pts) Let G be a connected Lie group. Let e be the identity element and U an open neighborhood of e in G. For each positive integer k, define

$$U^k := \{a_1 \cdots a_k \mid a_1, \ldots, a_k \in U\}.$$

Then show that $G = \displaystyle\bigcup_{k=1}^{\infty} U^k$.

END OF EXAM