Complex analysis

December 2014

1. (20points) Evaluate the following integrals and justify your answers.

   (a)
   $$\frac{1}{2\pi i}\int_{\gamma} ze^{1/z}dz,$$
   where γ is the unit circle $|z|=1$, oriented counterclockwise.

   (b)
   $$\int_{-\infty}^{\infty}\frac{x^2}{x^4+1}\,dx.$$

2. (10points) Evaluate
   $$\int_{-\infty}^{\infty}\frac{e^{px}}{1+e^x}\,dx,\quad 0<p<1.$$

3. (10points) Is there an analytic function in the open disk such that $|f(z)|=e^{|z|}$? Hint: Use the maximum modulus principle.

4. (20points) Prove that for real $\lambda>1$, there is a unique solution to the equation
   $$ze^{\lambda-z}=1$$
   in the open unit disk.

5. (20points) Let $D=\{z \mid |z|<1\}$. Suppose that $f: D\to\mathbb{C}$ satisfies $Ref(z)\geq 0$ for all z in D and suppose that f is analytic and not constant. Show that if $f(0)=1$ then
   $$|f(z)|\geq\frac{1-|z|}{1+|z|}.$$

6. (20points) Let $f(z)$ be an entire function such that $Ref(z)\leq M$ for some $M\in\mathbb{R}$, then show that f must be a constant.