1. Let A be an invertible square matrix. For some permutation P, Gaussian elimination gives LU-decomposition:

$$PA = LU.$$

There may also exist some other permutation $\widetilde{P}$ which gives another LU-decomposition

$$\widetilde{P}A = \widetilde{L}\widetilde{U}.$$

Prove that the set of absolute values of the pivots may be different, in other words,

$$\{|u_{kk}| : 1 \le k \le n\} \ne \{|\widetilde{u}_{kk}| : 1 \le k \le n\}.$$

(Hint : Find an example in 2×2-case.)

2. The so called Mercedes-Benz system in $\mathbb{R}^2$ is defined by

$$\{v_0, v_1, v_2\}, \quad \text{with} \quad v_k = \begin{bmatrix} \cos(2k\pi/3) \\ \sin(2k\pi/3) \end{bmatrix}.$$

Find the vectors $\{w_0, w_1, w_2\}$ in $\mathbb{R}^3$ orthogonal to each other and of the same length so that

$$v_k = P_{xy}(w_k), \quad (0 \le k \le 2) \quad \text{where} \quad P_{xy} \quad \text{is the projection onto xy-plane.}$$

(Consider $\mathbb{R}^2$ as a subspace of $\mathbb{R}^3$ with z-coordinate zero.)

3. Let A and B be $n \times n$ matrices for some $n \in \mathbb{N}$, satisfying

$$A = SBS^{-1}$$

for some matrix S and let

$$p(M) = \sum_{k=0}^{\ell} a_k M^k, \quad \text{with} \quad M^0 = I_n$$

for an $n \times n$ matrix M, where $p(x) = \sum_{k=0}^{\ell} a_k x^k$ is a degree ℓ polynomial with real coefficients.

(a) Prove that

$$p(A) = Sp(B)S^{-1}.$$

(b) When B is a diagonal matrix with $b_{kk} = \lambda_k$ for $1 \le k \le n$, prove that

$$\det(p(A)) = p(\lambda_1)p(\lambda_2)\cdots p(\lambda_n).$$