2014년 대수학II 박사자격시험(2014 Fall)

1. Prove that Aut$\mathbb{C}$ is an uncountable set,
where Aut$\mathbb{C}$ is the set of field automorphisms of $\mathbb{C}$.

2. Let F be the splitting field over $\mathbb{Q}$ of $f(x)=x^4-3$.
Find the number of subfields of F.

3. Let F$\geq$K be a field extension.
Prove that if $u\in F$ is purely inseparable over K and $v\in F$ is separable over K, then K(u,v)=K(u+v).

4. Let F be the algebraic closure of a finite field K.
   (a) Prove that $Aut_K F$ is an abelian group.
   (b) Prove that F does not have a subfield with finite index.

5. Let K be a finite extension of a field F with characteristic zero.
Show that there exists an element $\alpha\in$K such that K=F(α).

6. Suppose f(x) is an irreducible polynomial of order 5 over $\mathbb{Q}$, which has exactly 3 real roots in C.

Prove that the Galois group of a splitting field of f(x) over $\mathbb{Q}$ is the symmetric group on 5 letters.

7. Let K be a Galois extension of F with finite cyclic Galois group G generated by σ.

Prove that $\alpha\in K$ has the norm $N_{K/F}(\alpha)=1$ if and only if α is of the form $\alpha=\dfrac{\beta}{\sigma(\beta)}$ for some nonzero $\beta\in K$.

8. A field is F called perfect if every irreducible polynomial in F[x] is separable. Is the following statement true? Justify your answer.

"If $char\,F=p$, then F is perfect if and only if every element of F has a pth root which is also in F."

9. Prove that $\mathbb{Q}(\sqrt{2},\sqrt{3},\ldots,\sqrt{n})$=$\mathbb{Q}(\sqrt{2}+\ldots+\sqrt{n})$ for all natural number n$\geq$ 2.

10. Are the following statements true? Verify your answers.

(a) If K is a finite field, then for every n there exists an extension field F with $\dim_K F=n$.

(b) If K is an infinite field, then for every n there exists an extension field F with $\dim_K F=n$.

(c) For every $n\geq 1$, there exists a subfield K of $\mathbb{C}$ such that $\dim_K\mathbb{C}$ = n.