Algebra I Qualifying Exam

Jan, 2015

1. (a) (10 pts) For a finite abelian group $G = \{a_1, \ldots, a_s\}$, define its exponent by $\exp(G) := \text{lcm}(\text{ord}(a_1), \ldots, \text{ord}(a_s))$. Show that G is cyclic if and only if $\exp(G) = |G|$.

   (b) (10 pts) Let K be a field. Show that any finite subgroup of the multiplicative group K^* is cyclic.

2. (a) (10 pts) For an integer $x \in \mathbb{Z}$ and an odd prime p, define the Legendre symbol by

$$\left(\frac{x}{p}\right) := \begin{cases} 1 & \text{if } x \text{ is a square mod } p \\ -1 & \text{if } x \text{ is not a square mod } p. \end{cases}$$

   Prove that $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$.

   (b) (10 pts) Determine all integers that are prime in the ring $\mathbb{Z}[i]$ of Gaussian integers.

3. Consider the ring $R := \mathbb{Z}[i\sqrt{5}] = \{a + i\sqrt{5}b | a, b \in \mathbb{Z}\}$.

   (a) (10 pts) Consider the norm function $N : R \to \mathbb{Z}$ defined by $N(a + i\sqrt{5}b) = a^2 + 5b^2$. Show that N is multiplicative, i.e. $N(\alpha\beta) = N(\alpha)N(\beta)$. Use this to find all units of R.

   (b) (10 pts) Show that $6 = 2 \cdot 3$ and $6 = (1 + i\sqrt{5})(1 - i\sqrt{5})$ give two different irreducible facorizations of 6. Is R is a UFD? Is R a PID?

4. (15 pts) Consider the submodule M of $\mathbb{Z}^2$ generated by three vectors $\binom{6}{4}$, $\binom{2}{5}$ and $\binom{-2}{-3}$. Show that M is a free $\mathbb{Z}$-module by computing a free basis of M.

5. (a) (10 pts) Let p be an arbitrary given prime. Give an example of a nonabelian group G of order p^3.

   (b) (5 pts) For the group G constructed above, compute its center Z_G and order $|Z_G|$.

6. (10 pts) Show that the order of a finite field is a power of a prime.