Real Analysis

June 2014

1. (10pts) A real-valued function f defined on (a,b) is said to be convex if

$$f(\lambda x+(1-\lambda)y)\le \lambda f(x)+(1-\lambda)f(y)$$

whenever $a<x,y<b, 0<\lambda<1$. Prove that every convex function is continuous.

2. (15pts) If $\int_A f=0$ for every measurable subset A of a measurable set E, show that $f=0$ almost everywhere in E.

3. (15pts) Suppose $|E|<\infty$. If $f_k\to f$ almost everywhere in E and $|f_k|\le M$ for all $k=1,2,...$, show that $\int_E f_k\to\int_E f$.

4. (15pts) If $f\in L(0,1)$,

   (a) show that $x^k f(x)\in L(0,1)$ for $k=1,2,...$

   (b) $\lim_{k\to\infty}\int_0^1 x^k f(x)\,dx=0$.

5. (15pts) Suppose $f_k\to f$ in $L^p, 1\le p<\infty, g_k\to g$ pointwise. If $\|g_k\|_{L^\infty}\le M$ for all k, prove that $f_kg_k\to fg$ in L^p.

6. (15pts) For $f\in L(\mathbb{R})$, define the Fourier transform $\hat{f}$ of f by

$$\hat{f}(x)=\int_{-\infty}^{\infty} f(t)\,e^{-itx}dt,\quad x\in\mathbb{R}.$$

Show that if f and g belong to $L(\mathbb{R})$, then

$$(f*g)\hat{}\,(x)=\hat{f}(x)\,\hat{g}(x).$$

7. (15pts) Let $\phi(x), x\in\mathbb{R}^n$, be a bounded measurable function such that $\phi(x)=0$ for $|x|\ge 1$ and $\int_{\mathbb{R}^n}\phi(x)\;dx=1$. For $\varepsilon>0$, let $\phi_\varepsilon(x)=\varepsilon^{-n}\phi(x/\varepsilon)$. If $f\in L(\mathbb{R}^n)$, show that

$$\lim_{\varepsilon\to 0}(f*\phi_\varepsilon)(x)=f(x),\quad \text{almost everywhere in } \mathbb{R}^n.$$