1. (20points) Let $y_m = \int_0^1 \frac{x^m}{x+5}dx$, $m = 0, 1, 2, \cdots$. Derive a recurrence relation relating y_m to y_{m-1}.

   Consider the problem P to compute y_n by the recursion with initial term y_0:

$$y_0 \longrightarrow \boxed{P} \longrightarrow y_n$$

   Determine the condition number K_n of y_n at y_0 and find the limit $\lim_{\to\infty} K_n$. Discuss the numerical significance of your answers when $n \to \infty$.

2. (20points) For a function g, define $\|g\|^2 := \int_a^b |g''(x)|^2 dx$ if $\int_a^b |g''(x)|^2 dx < \infty$. Let S be the natural cubic spline function that interpolates a twice-continuously differentiable function f at knots $a = t_0 < t_1 \cdots < t - n = b$. Prove that

$$\int_a^b |S''(x)|^2 dx \leq \int_a^b |f''(x)|^2 dx.$$

   You may start with the term $\|f - S\|^2$.

3. (30points)
   (a) Describe the definition of the n-point Gaussian quadrature formula $O_n(f) = \sum_{i=1}^n w_i f(x_i)$, which approximates the integral $\int_{-1}^1 f(x)dx$.
   (b) Prove that the degree of $Q_n(f)$ is $2n - 1$. (Your answer should contains a reason why the degree of $Q_n(f)$ can not be greater than $2n - 1$.)
   (c) Show that $w_i > 0$ for each $i = 1, 2, \ldots n$.

4. (30points) Let

$$A = \begin{bmatrix} 4 & 1 & -1 & 1 \\ 1 & 4 & -1 & -1 \\ -1 & -1 & 5 & 1 \\ 1 & -1 & 1 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 3 \\ 4 \\ 5 \end{bmatrix}.$$

   (a) Prove that the matrix A is positive definite. Let $A_{4\times4}$ be a symmetric and strictly diagonally dominant matrix. Assume that all diagonal elements of $A_{4\times4}$ are positive. Prove that $A_{4\times4}$ is positive definite.
   (b) Find the matrix J for the Jacobi iterative method: $x^{(k+1)} = Jx^{(k)} + c$. Compute the first iteration $x^{(1)}$ with $x^{(0)} = 0$.
   (c) Prove that the Jacobi method converges.

The End