2014 PhD QE (Spring) **Probability & Statistics**

1. Let X be a real-valued random variable with density f (with respect to μ), and ψ be a non-negative Borel-measurable function on $\Re$. Show that

$$E[\psi(X)] = \int_{\Re} \psi(x) f(x) \mu(dx).$$

Does this equality holds for arbitrary Borel-measurable functions?

2. Let $X_1, X_2, \ldots$ be a sequence of independent Poisson random variables with $E(X_n) = \log\left(\frac{n^r}{n^r - 1}\right)$. Define events $A_n = \{X_n > 0\}$, $n = 1, 2, \ldots$.

   (a) Show that A_n occur infinitely often with probability if $r \leq 1$.

   (b) Is the statement in (a) still true for $r > 1$? If not, what would you conclude in this case?

3. Suppose that $X_n \to X$ almost surely. Furthermore, there is $M > 0$ such that $|X_n| \leq M$ for all $n = 1, 2, \ldots$. Prove that $X_n \to X$ in $\mathcal{L}_1$.

4. Let X and Y be random variables on a probability space $(\Omega, \mathcal{F}, P)$, and $X \leq Y$ almost surely. For any sub σ-field $\mathcal{G}$ of $\mathcal{F}$, show that

$$E(X|\mathcal{G}) \leq E(Y|\mathcal{G}), \text{ almost surely.}$$

5. Let $\{X_n; n = 0, 1, \ldots\}$ be a martingale, and let τ be stopping time such that $0 \leq \tau \leq K$ with probability 1 for some integer K. Verify that $E(X_\tau) = E(X_0)$.