**Differentiable Manifolds and Lie Groups**

July 2014

(1) (20pt.) Let $SO(n)$ be the group of $n \times n$ orthonormal real matrices of determinant 1.
  (a) Show that $SO(n)$ is a compact Lie group.
  (b) What is the dimension of $SO(n)$?

(2) (20pt.) Show that the 2-dimensional sphere is not a Lie group.

(3) (20pt.) Let $S^n$ be an $n$-dimensional sphere. Show that its $n$-th de Rham cohomology vector space is one-dimensional, i.e.,

$$H^n_{DR}(S^n) \cong \mathbb{R}.$$

(4) (20pt.) The space $\mathbb{R}^3$ has the force field given by the vector field

$$X = a\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) + b\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) + c\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$$

for real constants $a, b, c$. Suppose that at least one of the constants $a$, $b$, $c$ is non-zero. Find all the points in this space such that a particle placed at these points never moves.

(5) (20pt.) A Riemannian structure on a differentiable manifold $M$ is a smooth choice of an inner product $\langle \cdot, \cdot \rangle_p$ on each tangent space $T_pM$, smooth in the sense that whenever $X$ and $Y$ are smooth vector fields on $M$, then $\langle X, Y \rangle$ is a smooth function on $M$. Prove that every differentiable manifold has a Reimannian structure.