(1) Suppose that f and g are entire functions, and $|f(z)| \leq |g(z)|$ for every z. What conclusion can you draw? Justify your answer.

(2) Let Ω be a region. Suppose that $f \in H(\Omega)$, Γ is a cycle in Ω such that $\mathrm{Ind}_\Gamma(\alpha) = 0$ for all $\alpha \notin \Omega$, $|f(\zeta)| \leq 1$ for each ζ on the cycle Γ, and $\mathrm{Ind}_\Gamma(z) \neq 0$. Prove that $|f(z)| \leq 1$.

(3) Suppose that Ω is a simply connected region, and u is a real harmonic function in Ω. Prove that there exists a $f \in H(\Omega)$ such that $u = \mathrm{Re} f$.

(4) Let C be semicircle in the upper half plane indented at $z = 0$ with inner radius $\epsilon < 1$ and outer radius $R > 1$. Show that

$$\int_0^\infty \frac{\ln^2 x}{1+x^2} dx = \frac{\pi^3}{8} \quad \text{and} \quad \int_0^\infty \frac{\ln x}{1+x^2} dx = 0$$

by considering the contour integration

$$I = \int_C \frac{\log^2 z}{1+z^2} dz.$$

(5) If $u(z)$ is real harmonic and bounded in the punctured disk $0 < |z - z_0| < R$. Show that $\lim_{z \to z_0} u(z)$ exists.

(6) Let $f(z)$ be a complex polynomial. Prove that $f(z)$ is a constant if only if $f(z) = \sin z$ has infinitely many solutions.