Qualifying Exam

Algebra II

July 2014

(1) (20 points) Let F be a finite dimensional separable extension of an infinite field K.
   (a) Explain why the extension of K by F has only finitely many intermediate fields.
   (b) Show that $F = K(u)$ for some $u \in F$.

(2) (30 points) Let F_n be the splitting field of $x^n - 1$.
   (a) Let ζ is a primitive nth root of unity and let f be the minimal polynomial of ζ. Show that if p is a prime not dividing n, then ζ^p is a root of f.
   (b) Let $\Psi_n(x) = \prod_\zeta (x - \zeta)$ where ζ runs through all primitive nth roots of 1. Show that Ψ_n is monic, integral, and irreducible over $\mathbb{Q}$.
   (c) Determine the Galois group of F_n over $\mathbb{Q}$.

(3) (30 points) Compute the Galois group of L over K for the following extensions.
   (a) $K = \mathbb{Q}(e^{2\pi i/7})$; $L =$ the splitting field of $t^7 - 441$.
   (b) $K = \mathbb{Z}/p\mathbb{Z}$; $L =$ splitting field of $\prod_{i=1}^{p-1}(t^2 - i)$, where p is an odd prime.

(4) (20 points) Let $\zeta = e^{2\pi i/43}$ and

$$\alpha = \zeta + \zeta^4 + \zeta^{11} + \zeta^{16} + \zeta^{21} + \zeta^{35} + \zeta^{41}.$$

   (a) Compute the Galois group $\mathrm{Aut}_{\mathbb{Q}(\alpha)}\mathbb{Q}(\zeta)$.
   (b) Compute the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ by using the result of (a).

(5) (20 points)
   (a) Show that a UFD is integrally closed.
   (b) Find the integral closure $R = \mathbb{Q}[x, y]/\langle y^2 - x^3 \rangle$ in its quotient field, and explain why it is a Dedekind domain.