ALGEBRA 1 - QUALIFYING EXAM, SPRING, 2014

1. (15 points) Let K be a field. Show that any finite subgroup of the multiplicative group $K^* = K - \{0\}$ is always cyclic.

2. (25 points) Let G be a finite group.
(a) (5 points) Give a definition of nilpotent and solvable groups
(b) (10 points) Every finite p-group is nilpotent.
(c) (10 points) Show that a nilpotent group is solvable.

3. (20 points) Let R be a ring with identity. An element $e \in R$ is called an *idempotent* if $e^2 = e$. Assume that e is an idempotent in R and $er = re$ for all $r \in R$.
(a) (14 points) Show that Re and $R(1-e)$ are two-sided ideals of R and that $R \simeq Re \times R(1-e)$ as rings.
(b) (6 points) Prove that e and $1-e$ are identities for the subrings Re and $R(1-e)$ respectively.

4. (10 points) Let D be a unique factorization domain. When $f = \sum_{i=0}^{n} a_i x^i \in D[x]$ is a polynomial with coefficients in D, define $C(f)$ be a greatest common divisor of the coefficients $a_0, \cdots, a_n$. For $f, g \in D[x]$, show that $C(f \cdot g)$ and $C(f) \cdot C(g)$ are associates in D.

5. (20 points)
(a)(5 points) State the structure theorem for a finitely generated module over a principal ideal domain in terms of **invariant factors**.
(b)(5 points) State the structure theorem for a finitely generated module over a principal ideal domain in terms of **elementary divisors**.
(b)(10 points) If R is a nonzero commutative ring with identity and every submodule of every free R-module is free, then R is a a principal ideal domain.

6. (20 points) Let R and S be rings. Let $A = A_R$ be a right R-module and $C = C_S$ be a right S-module. Let $B = {}_R B_S$ be a bi-module (both a left R-module and a right S-module). Then show that there is an isomorphism of abelian groups

$$\alpha : \mathrm{Hom}_S(A \otimes_R B, C) \simeq \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)),$$

where Hom_S (respectively, Hom_R) means right S-module (respectively, R-module) homomorphisms. Note that the *right* S-module structure on $A \otimes_R B$ is given by: $(a \otimes b)s = a \otimes bs$ for $a \in A, b \in B, s \in S$ and the *right* R-module structure on $\mathrm{Hom}_S(B, C)$ is given by: $(gr)(b) = g(rb)$ for $r \in R, b \in B, g \in \mathrm{Hom}_S(B, C)$.