Real Analysis Qualifying Examination, Jan, 2014, POSTECH

(Justify all your work.. You may use the results of problem A to solve problem B.)

1. (19 points) Let m, m^* be the Lebesgue and Lebesgue outer measure respectively.

   a. Show that if $\text{dist}(E_1, E_2) > 0$, then $m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2)$.

   b. Show that every closed set is Lebesgue measurable.

2. (15 points) Let $f, \{f_k\}$ be measurable functions which are defined and finite a.e. in a set E. Suppose $\{f_k\}$ converges in measure on E: for every $\epsilon > 0$,

$$\mu\{x \in E: |f(x) - f_k(x)| > \epsilon\} \to 0 \quad \text{as } k \to \infty$$

Show that if all $f_k \geq 0$, then

$$\liminf_{k\to\infty} \int_E f_k d\mu \geq \int_E f d\mu$$

(You may use the fact: There exists a subsequence f_{k_j} that converges to f a.e. in E.)

3. (15 points) Let $\{E_k : k = 1,2,\cdots\}$ be measurable sets in R^n. Suppose $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Show that $\mu(A) = 0$ where $A = \{x \in R^n : x \in E_k \text{ for infinitely many } k\}$.

4. (15 points) Let $f \in L^1(R^n)$ and let $x \in R^n$ be a point: satisfying

$$A(r) := \frac{1}{r^n} \int_{|y| \leq r} |f(x - y) - f(y)|\, dy \to 0 \text{ as } r \to 0$$

Prove that $A(r)$ is continuous for $0 < r \leq 1$.

5. (15 points) Let $\{\phi_k : k = 1,2\cdots\}$ be an orthonormal system of $L^2(d\mu)$. Prove that $\{\phi_k : k = 1,2\cdots\}$ is complete if and only if $\|f\|_{L^2} = (\sum_{k=0}^{\infty} |\langle f, \phi_k \rangle|^2)^{1/2}$. (Being complete means: If $\langle f, \phi_k \rangle = 0$ for all k, then $f = 0$ in L^2 where $\langle f, g \rangle = \int f\bar{g}\, d\mu$.)

6. (21 points) Let $\{X, \Sigma, \mu\}$ be a finite measure space. Suppose ν is a measure defined on Σ. Suppose $\nu(A) = 0$ for any measurable set A with $\mu(A) = 0$. Define

$$\mathcal{F} := \left\{ f : \quad f \geq 0 \text{ is measurable}, \ \nu(E) \geq \int_E f d\mu, \qquad \text{for all } E \in \Sigma \right\}.$$

Since $0 \in \mathcal{F}$, $\mathcal{F}$ is nonempty. Let $s = \sup_{f \in \mathcal{F}} \int_X f d\mu$.

a. Prove that there exists a $g \in \mathcal{F}$ such that $s = \int_X g d\mu$. (Suggestion: Show first that if $f_1, f_2 \in \mathcal{F}$, then $\max\{f_1, f_2\} \in \mathcal{F}$)

Define $\nu_0(E) := \nu(E) - \int_E g d\mu$ so that ν_0 is a measure. Suppose $\nu_0(X) > 0$. Then there exists $\epsilon > 0$ such that $\nu_0(X) - \epsilon\mu(X) > 0$ since $\mu(X) < \infty$. Now by invoking Hahn decomposition,

there exists $P \in \Sigma$ such that for every $E \in \Sigma$, $\nu_0(E \cap P) - \epsilon\mu(E \cap P) \geq 0$,

b. Show that $g + \epsilon X_P \in \mathcal{F}$ where X_P is the characteristic function of P.

c. Show that $\mu(P) = 0$ and that $\nu_0(X) - \epsilon\mu(X) \leq 0$ which is a contradiction.