NUMERICAL ANALYSIS QE: JANUARY, 2014

Problem 1.(20 points) Let $\mathbf{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a twice differentiable vector function and let $\mathbf{F}(x,y) = [f(x,y), g(x,y)]$. Let $[x_*, y_*]$ be a zero of $\mathbf{F}(x,y)$, that is, $\mathbf{F}(x_*, y_*) = [0,0]$. To find the zero $\mathbf{x}_*$, (*i*) formulate the iterative method by the Newton's method, (*ii*) justify the convergence of the iterative method by specifying necessary conditions and (*iii*) give an example for the iteration of (*i*).

Problem 2.(20 points) (*i*) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a four times differentiable function. Let h be a positive number. Show the following in detail: There is a $\xi \in (x_0 - h, x_0 + h)$ such that

$$f''(x_0) = \frac{1}{h^2}[f(x_0 - h) - 2f(x_0) + f(x_0 + h)] - \frac{h^2}{12} f^{(4)}(\xi).$$

(*ii*) Let $x_0 = a, x_n = b, x_i = x_{i-1} + h$ for $i = 1, \cdots n-1$ with $h = (b-a)/n$ and consider the $(n+1)$-points Newton-Cotes formula:

$$\sum_{i=0}^{n} a_i f(x_i), \quad a_i = \int_{x_0}^{x_n} \prod_{j=0, j\neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)} dx.$$

Suppose n is an odd integer and $f \in C^{n+1}[a,b]$. Show that there is a $\xi \in (a,b)$ such that

$$\int_a^b f(x)dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!} \int_0^n t(t-1)\cdots(t-n)dt.$$

Problem 3.(20 points) Let A be an $n \times n$-matrix. Assume that there is an invertible matrix P such that $P^{-1}AP = D = diag[\lambda_1, \cdots, \lambda_n]$ is a diagonal matrix. Let B be any $n \times n$-matrix. Let λ be any eigenvalue of $A + B$. Then

$$\lambda \in \bigcup_{i=1}^{n} \{\mu \in \mathbb{C} : |\mu - \lambda_i| \leq \|P\|_\infty \|P^{-1}\|_\infty \|B\|_\infty,$$

where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of A. (Hint: Use the Gershgorin's Theorem)

Problem 4.(20points) Let A be the matrix defined by

$$A = \begin{bmatrix} 0 & 8 & 3 \\ 0 & 6 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{0.1}$$

(*i*) Give the definition of the singular value decomposition for any $m \times n$ matrix A. (*ii*) Find the singular value decomposition for the matrix A of (0.1). (*iii*) Find the pseudoinverse A^+ of A.

Problem 5.(20points) (*i*) Let A be a positive definite matrix of order n and $\langle\, , \rangle$ the inner product on $\mathbb{R}^n \times \mathbb{R}^n$. Show that the vector $\mathbf{x}^*$ is a solution to the equation $A\mathbf{x} = \mathbf{b} \in \mathbb{R}^n$ if and only if $\mathbf{x}^*$ minimizes $g(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle - 2\langle \mathbf{x}, \mathbf{b} \rangle$.

(*ii*) Suppose that A and B are $n \times n$ matrices so that $\|I - AB\| < 1$. Show that A and B are invertible, and $A^{-1} = B \sum_{k=0}^{\infty} (I - AB)^k$ and $B^{-1} = \sum_{k=0}^{\infty} (I - AB)^k A$.