2013 PhD QE (Fall) **Probability**

1. Let X be a real-valued random variable with density f (with respect to μ), and ψ be a non-negative Borel-measurable function on $\Re$.
Show that
$$E[\psi(X)] = \int_{\Re} \psi(x) f(x) \mu(dx).$$

2. Given a probability space $(\Omega, \mathscr{F}, P)$, prove the following statements:
   (a) For bounded random variables X and Y, $E[YE(X|\mathscr{G})] = E[XE(Y|\mathscr{G})]$ for any sub σ-field $\mathscr{G}$.
   (b) For sub σ-fields $\mathscr{G}_1 \subset \mathscr{G}_2$ and a random variable X such that $EX^2 < \infty$,
$$E[X - E(X|\mathscr{G}_1)]^2 \geq E[X - E(X|\mathscr{G}_2)]^2.$$

3. Let $X_1, X_2, \dots$ be independent positive random variables with $EX_n = 1$ for all n, and define $Y_n = X_1 X_2 \cdots X_n$.
   (a) Show that $\{Y_n\}$ converges to an integrable random variable Y almost surely.
   (b) Suppose X_n takes $\frac{1}{2}$ or $\frac{3}{2}$ with equal probabilities. Show that $E\left(\prod_{n=1}^{\infty} X_n\right) \neq \prod_{n=1}^{\infty} E(X_n)$.

4. Let $X_1, X_2, \dots$ be a sequence of random variables such that
   (i) $X_{n+m} - X_n$ is independent of $\mathscr{F}(X_1, \dots, X_n)$,
   (ii) the distribution of $X_{n+m} - X_n$ does not depend on n

   for every $n, m = 1, 2, \dots$. Verify that the statements (i) and (ii) hold even if n is replaced by a stopping time n^*.

5. Let W_t, $t \geq 0$, be a standard Brownian motion. Consider two parallel lines $y = a + rt$ and $y = -b + rt$, for positive a, b, r, and let T be the first time W_t meets either line, and $p(a, b, r)$ be the probability that the exit is through the upper barrier, i.e., $W_T = a + rT$.
   (a) Show that $Y_t(\theta) = \exp(\theta W_t - \frac{1}{2}\theta^2 t)$ is a martingale, and deduce that $E[Y_t(\theta)] = 1$ for all θ. Argue that for all θ, $E\left[\exp(\theta W_T - \frac{1}{2}\theta^2 T)\right] = 1$.
   (b) Obtain $p(a, b, r)$. What is the limiting value of $p(a, b, r)$ as $r \to 0$?