Qualifying Exam
Algebraic Topology
January 2014

(1) (a) Compute the cellular homology groups of the projective spaces $\mathbb{RP}^n$ and $\mathbb{CP}^n$.
(b) Compute the homology groups of

$$X = \{([x_0, x_1, x_2], [y_0, y_1]) \in \mathbb{CP}^2 \times \mathbb{CP}^1 \mid x_0 y_1 = x_1 y_0\}.$$

(Hint: Consider the natural projection map $\pi : X \to \mathbb{CP}^2$ and compare the homology of X with that of $\mathbb{CP}^2$.)

(2) (a) Let $X \subset S^n$ be homeomorphic to I^m, $m < n$, where I denotes the closed interval $[0, 1]$. Compute the homology groups of $S^n \setminus X$.
(b) Let $Y \subset S^n$ be homeomorphic to S^m, $m < n$. Use part (a) to show that

$$\widetilde{H}_k(S^n \setminus Y) = \begin{cases} \mathbb{Z}, & k = n - m - 1 \\ 0, & \text{else.} \end{cases}$$

(3) For a continuous map $f : S^n \to \mathbb{R}^{n+1} \setminus \{p\}$, define the $W(f, p)$ to be the degree of the associated map

$$\begin{array}{ccc} S^n & \longrightarrow & S^n \\ x & \mapsto & \dfrac{f(x) - p}{\|f(x) - p\|}. \end{array}$$

Let $f_1, f_2 : S^n \to \mathbb{R}^{n+1}$ satisfy the condition $\|f_1(x) - f_2(x)\| < \|f_1(x) - p\|$ for all $x \in S^n$. Show that $W(f_1, p) = W(f_2, p)$.

(4) Compute the homology groups of the following spaces.
(a) $\mathbb{R}^3 \setminus S^1$;
(b) The Möbius band;
(c) The complement of a Möbius band in $\mathbb{R}^3$.

(5) (a) Let $f : S^1 \to S^1$ be the double covering $f(z) = z^2$. Show that the mapping cone M_f is homeomorphic to the Möbius band.
(b) Show that there is no retraction from the Möbius band to its boundary circle. You may use part (b).