Lie Groups and Manifolds Qualifying Exam

1. Let M_n be the set of $n \times n$ with real entries and

$$O(n) = \{A \in M_n : A^t A = I\}.$$

Show that $O(n)$ is a manifold.

[10 marks]

2. Let $f : M \to N$ be a submersion of manifolds without boundary.

   (a) Show that if M is compact and N is connected, then f is onto.

   [10 marks]

   (b) Show that there is no submersion of a compact manifold without boundary into $\mathbb{R}^n$.

   [10 marks]

3. Let X and Y be smooth vector fields on a manifold M, and ϕ_t, ψ_s be the flows induced by X and Y respectively.

   (a) Show that

$$(\phi_s)_*[X, Y] = \lim_{t\to 0} \frac{1}{t}[(\phi_s)_*Y - (\phi_{s+t})_*Y]$$

   [10 marks]

   (b) Show that $\phi_t \circ \psi_s = \psi_s \circ \phi_t$ for every t, s if and only if $[X, Y] = 0$.

   [10 marks]

   (c) Show that a connected Lie group G is abelian if and only if its Lie algebra $\mathfrak{g}$ is abelian.

   [10 marks]

4. Equip $\mathbb{R}^6$ with the coordinates of $(x_1, x_2.x_3, y_1, y_2, y_3)$, and define the 1-form

$$\lambda = \sum_i x_i dy_i.$$

   (a) Compute $\omega := d\lambda$

   [10 marks]

(b) Compute $\omega \wedge \omega$.

[10 marks]

(c) Show that $\omega \wedge \omega$ is exact.

[10 marks]

5. Let M be a smooth manifold. Show that its tangent bundle TM is an orientable manifold.

[10 marks]