Qualifying Examination - Complex Analysis - 01/2014

1. Prove or disprove the following statements (Explain in short).

(a, 7pt) There exists a nonconstant entire function f such that $|f(z)| \leq \log|z|$ for $z \neq 0$.

(b, 7pt) Suppose that f is holomorphic in the region containing the closed unit disc and $|f(z)| < 1$ if $|z| = 1$. Then there is only one solution of the equation $f(z) = z$ in U.

(c, 8pt) There is a sequence of polynomials $\{P_n\}$ such that $P_n(0) = 1$ for $n = 1, 2, 3, \ldots$, but $P_n(z) \to 0$ for every $z \neq 0$ as $n \to \infty$.

2 (15pt). For $0 < a < 1$ evaluate $\int_0^\infty \frac{x^{a-1}}{1+x} dx$.

3. (a, 10pt) Construct an entire function f in an infinite product form such that f has only simple zeros at $\sqrt{n}$ for all positive integer n.

(b, 8pt) Does there exist a bounded holomorphic function f in the open right half plane such that f has only simple zeros at $\sqrt{n}$ for all positive integer n ? (Explain in short)

4. (a, 10pt) Let f be holomorphic in the open unit disc D with $f(0) = 1/2$. If $|f(z)| \leq 1$ for all $z \in D$, prove that $|f'(0)| \leq 3/4$.

(b, 5pt) Find all such $f(z)$ for which $f'(0) = 3/4$.

5(15pt). Let Ω be a simply connected region other than the plane with $0 \in \Omega$. Suppose that f is an one-to-one holomorphic function from Ω into the open unit disc D such that $f(\Omega)$ is a proper subset of D. Show that there is an one-to-one holomorphic function g from Ω into D such that $|f'(0)| < |g'(0)|$.

6(15pt). Find all real harmonic functions u in the unit disc D such that $u(0) = 0$ and u^2 is also harmonic in D.