Algebra II Qualifying Exam.

January 18, 2014

1. Let F, E, K be fields and T, S, R be commutative rings with identity. Prove:

(a) If F is an algebraic extension of E and E is an algebraic extension of K, then F is an algebraic extension of K.

(b) If T is an integral extension of S and S is an integral extension of R, then T is an integral extension of R.

2. Let $n \geq 1$ be an integer. Let K be a field and $K(x_1, \ldots, x_n)$ be the field of rational functions in $x_1, \ldots, x_n$, and $f_1, \ldots, f_n$ be elementary symmetric functions in $x_1, \ldots, x_n$. Show that $K(x_1, \ldots, x_n)$ is a Galois extension of $K(f_1, \ldots, f_n)$ with Galois group S_n.

3. Compute the Galois groups of the following polynomials over the field of rational numbers $\mathbf{Q}$: (a) $f = x^3 - 3x + 1$, (b) $f = x^4 - 2$, (c) $f = x^5 - 4x + 2$.

4. Let F be a finite dimensional extension of a finite field K. Show that the norm N_K^F and the trace T_K^F are surjective considered as maps $F \to K$.

5. Let A, B, D be $n \times n$ matrices. Show that $detX = detA \cdot detD$ where X is a $2n \times 2n$ matrix given by

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

6. The following result is known as weak Nullstellensatz: If F is an algebraically closed field of a field K and I is a proper ideal of $K[x_1, \ldots, x_n]$, then the affine variety $V(I)$ defined by I in F^n is nonempty. Assume the weak Nullstellensatz and prove Hilbert's Nullstellensatz.

7. Let $g_n(x)$ be the nth cyclotomic polynomial over the field of rational numbers $\mathbf{Q}$.

(a) Compute $g_{12}(x)$.

(b) Show that $g_n(x)$ is irreducible over $\mathbf{Q}$.

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