Algebra I Qualifying Exam.

January 17, 2013

1. Classify all groups of order ≤ 15 up to isomorphism.

2. Let D be a regular dodecahedron in three dimensional Euclidean space. Recall that a symmetry of D is a bijective map $D \to D$ that preserves distances and maps adjacent vertices onto adjacent vertices. Compute the order of the group of symmetries of D.

3. Answer the following problems:

(a) Show that every nilpotent group is solvable.

(b) Give an example of solvable group which is not nilpotent.

4. Answer the following problems:

(a) Show that the subring $\{a + bi | a, b \in \mathbf{Z}\}$ of complex numbers is a Euclidean domain.

(b) Show that every Euclidean domain is a principal ideal domain.

(c) Show that the subring $\{a + b\frac{1+\sqrt{19}i}{2} | a, b \in \mathbf{Z}\}$ of complex numbers is a principal ideal domain that is not a Euclidean domain.

5. Let R be the ring of $n \times n$ matrices over a division ring D, where $n \geq 1$ is an integer. Show that R has no proper ideals.

6. Give a sketch of proof of the following result which plays a central role in investigation of the tensor product of free modules: Let R be a ring with identity. If A is a unitary right R-module and F is a free left R-module with basis Y, then every element u of $A \otimes_R F$ may be written uniquely in the form $u = \sum_{i=1}^{n} a_i \otimes y_i$, where $a_i \in A$ and the y_i are distinct elements of Y.

The End