Real Analysis, Ph.D. Qualifying Examination, June 2013, Postech

Justify all your work. Do not interpret a problem so that it becomes trivial.

Notation. Let $dx = dm$ (or dy) denote Lebesgue measure in $\mathbb{R}^d$, and let $|E|$ be Lebesgue measure of a set E in $\mathbb{R}^d$.

1. (9+9 points) (a) Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a linear mapping and let E be a measurable set in $\mathbb{R}^d$. Show that

$$|T(E)| = |\det T| \cdot |E|.$$

(You may assume this holds when $E = I$ is a (d-dimensional) rectangle in $\mathbb{R}^d$.)

(b) Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a nonsingular linear mapping. Suppose f is a measurable function on a measurable set E in $\mathbb{R}^d$ and that $\int_E f(y)\,dy$ exists. Show that the following change of variables formula holds:

$$\int_E f(y)\,dy = |\det T| \int_{T^{-1}(E)} f(Tx)\,dx.$$

2. (15 points) Suppose that f_n converges in measure to f on $\mathbb{R}^d$. (Namely, for every $\varepsilon > 0$, $|\{x \in \mathbb{R}^d : |f_n(x) - f(x)| > \varepsilon\}| \to 0$ as $n \to \infty$.) Show that there exists a subsequence f_{n_k} which converges to f a.e.

3. (9+9 points) (a) Let f, g, f_n, $g_n \in L^1(\mathbb{R}^d)$, $|f_n| \le g_n$, and suppose that $f_n \to f$ a.e. and $g_n \to g$ a.e. If $\int g_n \to \int g$, show that $\int f_n \to \int f$.

(b) Let $1 \le p < \infty$. Assume that f_n, $f \in L^p(\mathbb{R}^d)$, and $f_n \to f$ a.e. If $\|f_n\|_p \to \|f\|_p$, then show that $\|f_n - f\|_p \to 0$, as $n \to \infty$.

4. (16 points) Let $\{f_n\}$ be a sequence of real-valued measurable functions on $\mathbb{R}^d$ which converges to a function f pointwise. Assume that $|f_n(x)| \le g(x)$ for all $x \in \mathbb{R}^d$, $n \ge 1$, where $g \in L^1(\mathbb{R}^d)$. Show that for every $\varepsilon > 0$, there exists a measurable set $E \subset \mathbb{R}^d$ with $|E| < \varepsilon$ such that $f_n \to f$ uniformly on E^c.

(Suggestion: consider $E_{n,k} = \bigcup_{j=n}^{\infty} \{x \in \mathbb{R}^d : |f_j(x) - f(x)| > 1/k\}$, and imitate the proof of Egoroff's theorem.)

5. (16 points) For $f \in L^1(\mathbb{R}^d)$ and $r > 0$, define

$$A_r(f)(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy$$

where $B(x,r)$ denotes the open ball of radius r centered at x. Show that $A_r(f) \to f$ a.e. on $\mathbb{R}^d$, as $r \to 0^+$. (This is a version of the Lebesgue differentiation theorem. Do not deduce it from some version of that theorem.)

6. (17 points) Let $1 \le n_1 < n_2 < \cdots$ be a sequence of integers, and define E to be the set of all $x \in [0, 2\pi)$ such that $\cos(n_k x)$ converges (to some $f(x)$) as $k \to \infty$. (Explain briefly why E is measurable.) Show that $|E| = 0$.

(Suggestion: consider the limits of $\int_A \cos(n_k x)\,dx$ and $\int_A [\cos(n_k x)]^2\,dx = (1/2) \int_A [1 + \cos(2 n_k x)]\,dx$ for some subsets A of E.)