1. (20 points) Consider the formula $x(p) = -p + \sqrt{p^2+1}$ to compute a root of the equation

$$x^2 + 2px - 1 = 0.$$

(a) Find the condition number $cond(x)(p)$.

(b) For what value p is the root-finding problem by the formula $x(p)$ well-conditioned?

2. (20 points) Let $L_i(x)$ be the Lagrange polynomials for pairwise different support abscissas $x_0, \ldots, x_n$, and let $c_i := L_i(0)$. Show that

(a) $\sum_{i=0}^{n} L_i(x) \equiv 1$ for every integer $n \geq 1$.

(b) $\sum_{i=0}^{n} c_i x_i = 0$ for every integer $n \geq 1$.

3. (20 points) Let $\Delta = \{x_0 < x_1 < x_2 < x_3 < x_4\} = \{-2 < -1 < 0 < 1 < 2\}$. Find the natural cubic spline $S_\Delta(x)$ such that

$$S_\Delta(-2) = S'_\Delta(-2) = S''_\Delta(-2) = 0,$$
$$S_\Delta(0) = 1,$$
$$S_\Delta(2) = S'_\Delta(2) = S''_\Delta(2) = 0.$$

4. (20 points)

(a) Derive the three-point Gaussian quadrature rule $Q(f)$

$$\int_{-1}^{1} f(x)dx \approx Q(f) := w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3).$$

(b) With the result of (a), derive the three-point Gaussian quadrature rule $Q^h(f)$ over the interval $[a, a+h]$

$$\int_{a}^{a+h} f(x)dx \approx Q^h(f) = w_1^h f(x_1^h) + w_2 f(x_2^h) + w_3^h f(x_3^h).$$

5. (20 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}.$$

(a) By using the partial pivoting, we want to find the LU-decomposition

$$LU = PA,$$

where L is a lower triangular matrix with 1's on the diagonal and U is an upper triangular matrix, and P is a permutation matrix. Find the matrices L, U, P.

(b) Find the condition number $cond_\infty(A)$ of the matrix A.

6. (20 points) We shall use Gerschgorin's theorem which gives an inclusion region of all eigenvalues of a matrix $A_{n\times n}$.

(a) State in detail Gerschgorin's theorem and prove it.

(b) Applying Gerschgorin's theorem to the matrix A in (5), find a region including all eigenvalues of the matrix A and sketch it on the complex plane.

The End